9

EVALUATION OF LINEAR AND ANGULAR MOMENTUM INTEGRALS

Following Page and Adams (1945), equations are derived for the linear momentum and angular momentum imparted to the field by two interacting electric charges. Results are generalized to include the interaction between any pair of multivectors. We also determine torque density from which one may obtain reaction forces on the multivectors from the time rate of charge of the angular momenta. Details are given for evaluation of pertinent integrals.

The forces we observe in this chapter and in Chapter 10 are effectively drag forces acting on the particles (planets) by virtue of the reaction force experienced when they interact and also when they move as a single particle–described by Feynman (1977). The drag force slows the motion so that the planet drops to a lower orbit during each revolution. However its velocity in the lower orbit is larger than that required for equilibrium. Thus, the extreme point in the orbit, called the perihelion, is reached earlier than an equilibrium value. This is the origin of the so-called perihelion advance. We list this correction as item 2 in the summary provided in Chapter 2 where the same effect is obtained by adding $3v^2/c^2$ to G.

All corrections applied to Mercury hold for the other planets and the sun but have an insignificant effect so that Newtonian mechanics is sufficient to describe their motion.

For electromagnetism and gravity, the drag force originates from the particles themselves. Other than that qualification, the phenomenon is similar to that of the "Satellite Paradox" (Blake 1959). As a satellite enters the atmosphere, it experiences atmospheric resistance, loses energy, and drops to a lower orbit. However, its velocity in the lower orbit is greater than that required for equilibrium in orbit and it thereby completes the orbit in less time than the preceding revolution. In this case the drag force is external to the satellite. Other than that, the phenomenon is similar to gravitational drag.

9.1 Linear Momentum Imparted to the Field by $q_s\mathbf{v}_s$ and $q_t\mathbf{v}_t$

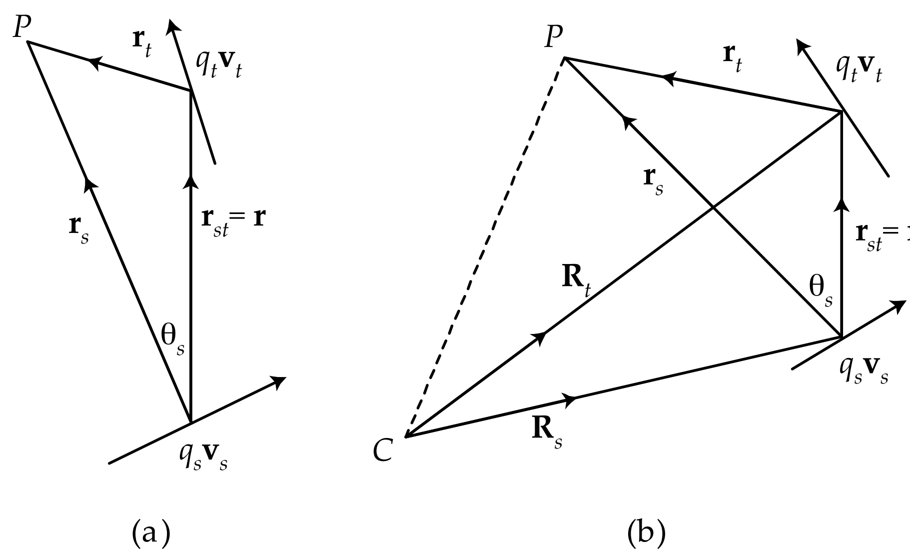

Fig. 9.1

We now evaluate the integrals involved in calculating the momentum generated by $q_s\mathbf{v}_s$ and $q_t\mathbf{v}_t$. The electromagnetic linear momentum, $\mathbf{g}_\ell$ per unit volume imparted to the field at point P by the motion of the two moving charges $q_s\mathbf{v}_s$ and $q_t\mathbf{v}_t$ is given by

$$\begin{aligned}\mathbf{g}_\ell &= \varepsilon_0\left(\mathbf{E}_t + \mathbf{E}_s\right) \times \left(\mathbf{B}_t + \mathbf{B}_s\right)\\ \mathbf{g}_\ell &= \varepsilon_0\left(\mathbf{E}_t \times \mathbf{B}_t + \mathbf{E}_t \times \mathbf{B}_s + \mathbf{E}_s \times \mathbf{B}_t + \mathbf{E}_s \times \mathbf{B}_s\right)\end{aligned}$$

In this section, we are interested only in the interaction of one current element with the other and therefore retain only the two middle terms in the above. Thus the mutual momentum density at point P is given by the sum

$$\mathbf{g}_\ell = \varepsilon_0\mathbf{E}_s \times \mathbf{B}_t + \varepsilon_0\mathbf{E}_t \times \mathbf{B}_s \tag{9.1}$$

The first term when integrated over all space and then differentiated with respect to the time gives the force of $q_s\mathbf{v}_s$ on $q_t\mathbf{v}_t$. We will call it the "indirect" force or Poynting force of q_s on q_t, that is,

$$\mathbf{F}_{ts}^{\varepsilon_0 \frac{d}{dt}\int(\mathbf{E}_s\times\mathbf{B}_t)d\tau} = d\mathbf{G}_\ell^{\varepsilon_0(\mathbf{E}_s\times\mathbf{B}_t)}/dt = \text{force on } q_t\mathbf{v}_t \text{ (reactive force on } q_s\mathbf{v}_s) \tag{9.2}$$

The same procedure applied to the second term gives the indirect force of $q_t\mathbf{v}_t$ on $q_s\mathbf{v}_s$.

$$\mathbf{F}_{st}^{\varepsilon_0 \frac{d}{dt}\int(\mathbf{E}_t\times\mathbf{B}_s)d\tau} = d\mathbf{G}^{\varepsilon_0(\mathbf{E}_t\times\mathbf{B}_s)}/dt = \text{force on } q_s\mathbf{v}_s \text{ (reactive force on } q_t\mathbf{v}_t) \tag{9.3}$$

The total linear momentum contributed to the field by the terms in Eq. (9.1) is obtained by integrating the momentum densities, Eqs. (9.2) and (9.3), over all space. We label the total linear momenta $\mathbf{G}_{ts\ell}$ and $\mathbf{G}_{st\ell}$, respectively. The result, after integration, is:

$$\mathbf{G}_{ts\ell} = \int \varepsilon_0\left(\mathbf{E}_s\times\mathbf{B}_t\right)d\tau = \frac{q_sq_t}{4\pi\varepsilon_0c^2}\frac{1}{2}\left[\frac{\mathbf{v}_t}{r}+\frac{\left(\mathbf{v}_t\cdot\mathbf{r}_{st}\right)\mathbf{r}_{st}}{r^3}\right] \tag{9.4}$$

$$\mathbf{G}_{st\ell} = \int \varepsilon_0\left(\mathbf{E}_t\times\mathbf{B}_s\right)d\tau = \frac{q_tq_s}{4\pi\varepsilon_0c^2}\frac{1}{2}\left[\frac{\mathbf{v}_s}{r}+\frac{\left(\mathbf{v}_s\cdot\mathbf{r}_{st}\right)\mathbf{r}_{st}}{r^3}\right] \tag{9.5}$$

A calculation of the total angular momentum, $\mathbf{G}_a$, about an arbitrary point C, the position vector of which measured from $q_s\mathbf{v}_s$ is $\mathbf{R}_s$ and measured from $q_t\mathbf{v}_t$ is $\mathbf{R}_t$, gives, as shown later:

$$\mathbf{G}_a = \frac{q_sq_t}{4\pi\varepsilon_0c^2}\left[\mathbf{R}_t\times\overbrace{\frac{1}{2}\left(\frac{\mathbf{v}_s}{r}+\frac{\left(\mathbf{v}_s\cdot\mathbf{r}_{st}\right)r_{st}}{r^3}\right)}^{\text{reactive force on } q_s}+\mathbf{R}_s\times\overbrace{\frac{1}{2}\left(\frac{\mathbf{v}_t}{r}+\frac{\left(\mathbf{v}_t\cdot\mathbf{r}_{st}\right)\mathbf{r}_{st}}{r^3}\right)}^{\text{reactive force on } q_t}\right] \tag{9.6}$$

That is,

$$\mathbf{G}_a = \frac{q_sq_t}{4\pi\varepsilon_0c^2}\left[\mathbf{R}_t\times\varepsilon_0\int\left(\mathbf{E}_t\times\mathbf{B}_s\right)d\tau+\mathbf{R}_s\times\varepsilon_0\int\left(\mathbf{E}_s\times\mathbf{B}_t\right)d\tau\right] \tag{9.7}$$

The first term in Eq. (9.6) gives the total angular momentum in the field associated with lever arm $\mathbf{R}_t$. The second term gives the portion of the total angular momentum about C associated with the lever arm $\mathbf{R}_s$. To quote Page and Adams (1945), "we see that the portion of the linear momentum involving the velocity $q_s\mathbf{v}_s$ of particle q_s is to be considered as located at $q_t\mathbf{v}_t$. Likewise, the portion of the angular momentum associated with particle $q_t\mathbf{v}_t$ is to be considered as located at the position of $q_s\mathbf{v}_s$."

The total reactive torque $\mathbf{T}$ about P delivered to the field is

$$\mathbf{T} = \frac{d\mathbf{G}_a}{dt} = \mathbf{R}_t \times \text{reactive force on } q_s\mathbf{v}_s \text{ plus } \mathbf{R}_s \times \text{reactive force on } q_t\mathbf{v}_t \tag{9.8}$$

$$= \mathbf{R}_t \times \left[\frac{d}{dt}\int \varepsilon_0 \left(\mathbf{E}_t \times \mathbf{B}_s\right) d\tau + \mathbf{R}_s \times \frac{d}{dt}\int \varepsilon_0 \left(\mathbf{E}_s \times \mathbf{B}_t\right) d\tau\right] \tag{9.9}$$

$$= \mathbf{R}_t \times \overbrace{\frac{d}{dt}\mathbf{G}_\ell^{\varepsilon_0(\mathbf{E}_t\times\mathbf{B}_s)}}^{\text{reactive force on } q_s\mathbf{v}_s} + \mathbf{R}_s \times \overbrace{\frac{d}{dt}\mathbf{G}_\ell^{\varepsilon_0(\mathbf{E}_s\times\mathbf{B}_t)}}^{\text{reactive force on } q_t\mathbf{v}_t} \tag{9.10}$$

$$\mathbf{T} = \frac{q_s q_t}{4\pi\varepsilon_0 c^2}\left[\mathbf{R}_t \times \frac{d}{dt}\frac{1}{2}\left(\frac{\mathbf{v}_s}{r} + \frac{\left(\mathbf{v}_s \cdot \mathbf{r}_{st}\right)\mathbf{r}_{st}}{r^3}\right) + \mathbf{R}_s \times \frac{d}{dt}\frac{1}{2}\left(\frac{\mathbf{v}_t}{r} + \frac{\left(\mathbf{v}_t \cdot \mathbf{r}_{st}\right)\mathbf{r}_{st}}{r^3}\right)\right] \tag{9.11}$$

For evaluation of the time derivative terms, see Section 10.8.

Eq. (9.8) appears counter-intuitive since when dealing with direct forces we cross the lever-arm at q_s into the force experienced by q_s to obtain the torque about P (Page and Adams 1945). In examining Eq. (9.10) we must remember that the force on q_s is not generated directly by q_t but is the result of evaluating the Poynting vector which involves both q_t and q_s.

The other alternative $\mathbf{R}_t \times \dfrac{d}{dt}\mathbf{G}_\ell^{\varepsilon_0(\mathbf{E}_s\times\mathbf{B}_t)}$ would be equally counter-intuitive, in that the indirect torque, which involves both q_s and q_t, would be calculated the same way we calculated the direct torque, that is, the direct force on q_t times the lever arm from C to q_t. We adopt Eq. (9.10) which says that the force on q_s, which is generated jointly by q_s and q_t, must be multiplied by the lever arm as measured from the fulcrum to the position of q_t .

Following are details.

From Fig. 9.1, the magnetic fields and the electric fields at point P generated by $q_s\mathbf{v}_s$ and $q_t\mathbf{v}_t$ are

$$\mathbf{B}_s = \frac{q_s}{4\pi\varepsilon_0 c^2}\frac{\mathbf{v}_s \times \mathbf{r}_s}{r_s^3} = \frac{q_s\mathbf{v}_s \times \mathbf{r}_s}{\left(4\pi\varepsilon_0\right)c^2 r_s^3}, \qquad \mathbf{B}_t = \frac{q_t}{\left(4\pi\varepsilon_0\right)c^2}\frac{\mathbf{v}_t \times \mathbf{r}_t}{r_t^3}$$

$$\mathbf{E}_s = \frac{q_s}{4\pi\varepsilon_0}\frac{\mathbf{r}_s}{r_s^3}, \qquad \mathbf{E}_t = \frac{q_t}{4\pi\varepsilon_0}\frac{\mathbf{r}_t}{r_t^3} \tag{9.12}$$

Thus the mutual linear momentum densities are

$$\mathbf{g}_{ts\ell} = \varepsilon_0\mathbf{E}_s \times \mathbf{B}_t = \frac{q_s q_t}{\left(4\pi\right)^2 \varepsilon_0 c^2}\frac{\mathbf{r}_s \times \left(\mathbf{v}_t \times \mathbf{r}_t\right)}{r_s^3 r_t^3} = k\frac{\mathbf{r}_s \times \left(\mathbf{v}_t \times \mathbf{r}_t\right)}{r_s^3 r_t^3} \tag{9.13}$$

$$\mathbf{g}_{st\ell} = \varepsilon_0\mathbf{E}_t \times \mathbf{B}_s = \frac{q_t q_s}{\left(4\pi\right)^2 \varepsilon_0 c^2}\frac{\mathbf{r}_t \times \left(\mathbf{v}_s \times \mathbf{r}_s\right)}{r_t^3 r_s^3} = k\frac{\mathbf{r}_t \times \left(\mathbf{v}_s \times \mathbf{r}_s\right)}{r_s^3 r_t^3} \tag{9.14}$$

where we have used $\mu_0\varepsilon_0 = 1/c^2$ and let $k = q_s q_t / (4\pi)^2 \varepsilon_0 c^2$.

When forming $\mathbf{E}_s \times \mathbf{B}_t$, it is only necessary to use the static Coulomb law, Eq. (9.8) for $\mathbf{E}_s$ in order that $\mathbf{E}_s \times \mathbf{B}_t$ be given to second order in $1/c$ since $\mathbf{B}_t$ is $1/c^2$ smaller than $\mathbf{E}_s$. When calculating the direct force of $\mathbf{E}_s$ on q_s, it was necessary to evaluate $\mathbf{E}_s$ to higher order to obtain an accuracy of $1/c^2$. The calculations herein will be carried out only to second order, that is, to $1/c^2$.

The following section shows the integration details leading to Eq. (9.4).

9.2 Evaluation of Integrals to Obtain Equations (9.4) and (9.5)

From Fig. 9.1

$$r_t^2 = r_s^2 + r_{st}^2 - 2r_{st} r_s \cos\theta_s$$

$\mathbf{r} \equiv \mathbf{r}_{st}$ is the distance from q_s to q_t. $\mathbf{r}_{st}$ is not a variable in this integration and is effectively constant in magnitude and direction (Page and Adams 1945). For constant r_s, $r_t dr_t = r_{st} r_s \sin\theta_s d\theta_s$. Therefore the volume element $d\tau = r_s^2 \sin\theta_s dr_s d\theta_s d\varphi$ may be written in terms of r_s, r_t, and φ as variables. Eliminating $\sin\theta_s d\theta_s = r_t dr_t / r_{st} r_s$, the volume element may be written $d\tau = (r_s r_t / r_{st})\, dr_s dr_t d\varphi$. For $r_s < r$, the limits of r_t are $r_{st} - r_s$ and $r_{st} + r_s$. For $r_s > r_{st}$, r_t goes from $r_s - r_{st}$ to $r_s + r_{st}$.

$$\begin{aligned} \sin\theta_s d\theta_s &= \frac{r_t dr_t}{r_{st} r_s} r_s \\ d\tau &= r_s^2 \sin\theta_s dr_s d\theta_s d\varphi = \frac{r_s dr_s d\theta r_t dr_t}{r_{st}} = \frac{r_s dr_s r_t dr_t d\varphi}{r_{st}} \end{aligned}$$

We now evaluate the integral of Eq. (9.4). The integrand is the linear momentum density, $\mathbf{g}_{ts\ell}$. The same for Eq. (9.5) is obtained from the former by interchanging the subscripts s and t and will not be considered further.

We will use

$$\begin{aligned} \mathbf{r}_s \times (\mathbf{v}_t \times \mathbf{r}_t) &= \mathbf{v}_t (\mathbf{r}_s \cdot \mathbf{r}_t) - \mathbf{r}_t (\mathbf{r}_s \cdot \mathbf{v}_t) \quad (9.15) \\ \mathbf{r}_t \times (\mathbf{v}_s \times \mathbf{r}_s) &= \mathbf{v}_s (\mathbf{r}_s \cdot \mathbf{r}_t) - \mathbf{r}_s (\mathbf{r}_t \cdot \mathbf{v}_s) \quad (9.16) \end{aligned}$$

$$\mathbf{g}_{ts\ell} = k\frac{\mathbf{r}_s \times (\mathbf{v}_t \times \mathbf{r}_t)}{r_s^3 r_t^3} = k\frac{\mathbf{v}_t (\mathbf{r}_s \cdot \mathbf{r}_t) - \mathbf{r}_t (\mathbf{v}_t \cdot \mathbf{r}_s)}{r_s^3 r_t^3} \quad (9.17)$$

Summary

$$\mathbf{G}_{ts\ell} = \int \mathbf{g}_{ts\ell} d\tau, \qquad \int \mathbf{F}_{ts}^{\varepsilon_0 \frac{d}{dt}(\mathbf{E}_s \times \mathbf{B}_t)} = d\mathbf{G}_{ts\ell}/dt, \qquad k = q_t q_s / (4\pi)^2 \varepsilon_0 c^2 \quad (9.18)$$

Consider the integral of the first term in Eq. (9.17). Call it $\mathbf{G}_{ts\ell 1}$.

$$\mathbf{G}_{ts\ell 1} = k\int \frac{\mathbf{v}_t\left(\mathbf{r}_s \cdot \mathbf{r}_t\right)}{r_s^3 r_t^3} d\tau \qquad d\tau = \frac{r_s r_t\, dr_s dr_t d\varphi}{r_{st}} \tag{9.19}$$

$$\begin{aligned} d\tau &= \left(r_s r_t /r_{st}\right) dr_s dr_t d\varphi \\ 2r_{st} r_s \cos\theta_s &= r_{st}^2 + r_s^2 - r_t^2 \end{aligned}$$

$$\begin{aligned} \mathbf{r}_s \cdot \mathbf{r}_t &= \mathbf{r}_s \cdot \left(\mathbf{r}_s - \mathbf{r}_{st}\right) \\ &= r_s^2 - \mathbf{r}_s \cdot \mathbf{r}_{st} = r_s^2 - \left(r_{st}^2 + r_s^2 - r_t^2\right)/2 = \left(r_s^2 + r_t^2 - r_{st}^2\right)/2 \end{aligned} \tag{9.20}$$

When Eq. (9.19) is integrated over $d\varphi$, that is, 0 to 2π, we have left

$$\begin{aligned} \mathbf{G}_{ts\ell 1} &= k2\pi\mathbf{v}_t \iint \frac{\mathbf{r}_t \cdot \mathbf{r}_s}{r_t^3 r_s^3} d\tau = k\pi\mathbf{v}_t \iint \frac{\left(r_s^2 + r_t^2 - r_{st}^2\right) r_s r_t}{r_t^3 r_s^3 r_{st}} dr_s dr_t \\ &= \frac{k\pi\mathbf{v}_t}{r_{st}} \iint \frac{\left(r_s^2 + r_t^2 - r_{st}^2\right)}{r_s^2 r_t^2} dr_s dr_t \end{aligned} \tag{9.21}$$

Explicitly,

$$\begin{aligned} \mathbf{G}_{ts\ell 1} &= \frac{k\pi\mathbf{v}_t}{r_{st}} \left[\int_0^{r_{st}} \frac{dr_s}{r_s^2} \int_{r_{st}-r_s}^{r_{st}+r_s} \left(\frac{r_s^2 - r_{st}^2}{r_t^2} + 1\right) dr_t \right. \\ &\quad \left. + \int_{r_{st}}^{\infty} \frac{dr_s}{r_s^2} \int_{r_s - r_{st}}^{r_{st}+r_s} \left(\frac{r_s^2 - r_{st}^2}{r_t^2} + 1\right) dr_t \right] \end{aligned} \tag{9.22}$$

$$\begin{aligned} \int_{r_{st}-r_s}^{r_{st}+r_s} \frac{dr_t}{r_t^2} &= -\left[\frac{1}{r_t}\right]_{r_{st}-r_s}^{r_s+r_{st}} = \frac{-1}{r_{st}+r_s} + \frac{1}{r_{st}-r_s} = \frac{2r_s}{r_{st}^2 - r_s^2}, \\ \int_{r_{st}-r_s}^{r_{st}+r_s} dr_t &= 2r_s, \qquad \int_{r_s-r_{st}}^{r_{st}+r_s} dr_t = 2r_{st} \\ \int_{r_s-r_{st}}^{r_s+r_{st}} \frac{dr_t}{r_t^2} &= \frac{2r_{st}}{r_s^2 - r_{st}^2} \end{aligned}$$

Thus,

$$
\begin{aligned}
\mathbf{G}_{ts\ell 1} &= \frac{k\pi \mathbf{v}_t}{r_{st}} \left[\int_0^{r_{st}} \frac{dr_s}{r_s^2} \left[\left(r_s^2 - r_{st}^2 \right) \frac{2r_s}{r_{st}^2 - r_s^2} + 2r_s \right] \right. \\
&\qquad \left. + \int_{r_{st}}^{\infty} \frac{dr_s}{r_s^2} \left[- \frac{\left(r_s^2 - r_{st}^2 \right) 2r_{st}}{r_s^2 - r_{st}^2} + 2r_{st} \right] \right] \\
&= \frac{k\pi \mathbf{v}_t}{r_{st}} \left[\int_0^{r_{st}} \frac{dr_s}{r_s^2} [0] + 4r_{st} \int_{r_{st}}^{\infty} \frac{dr_s}{r_s^2} \right] = 4k\pi \mathbf{v}_t \left[- \frac{1}{r_s} \right]_{r_{st}}^{\infty} = \frac{4k\pi \mathbf{v}_t}{r_{st}} \\
\mathbf{G}_{ts\ell 1} &= \int \int \mathbf{g}_{ts\ell 1} d\tau = \frac{4\pi k}{r_{st}} \mathbf{v}_t = \frac{4\pi k}{r_{st}} \frac{q_t q_s \mathbf{v}_t}{(4\pi)^2 \varepsilon_0 c^2} \\
\mathbf{G}_{ts\ell 1} &= \frac{q_t q_s}{4\pi \varepsilon_0 c^2} \frac{\mathbf{v}_t}{r_{st}} = \frac{\mu_0}{4\pi} \frac{\mathbf{v}_t}{r_{st}}
\end{aligned}
\tag{9.23}
$$

We repeat the derivation of the integral of $\mathbf{g}_{ts\ell\ell 1}$using dimensionless variables[1], that is, express distances in multiples of r_{st}.
Starting with Eq. (9.20)

$$
\begin{aligned}
(\mathbf{r}_s \cdot \mathbf{r}_t) &= r_{st}^2 \left(\frac{\rho_s^2 + \rho_t^2 - 1}{2} \right) \\
\mathbf{G}_{ts\ell 1} &= k \int \int \frac{\mathbf{v}_t \left(\mathbf{r}_s \cdot \mathbf{r}_t \right)}{r_s^3 r_t^3} d\tau \\
r_s &= r_{st} \rho_s \qquad r_t = r_{st} \rho_t \\
d\tau &= \frac{r_s r_t dr_s r_t d\varphi}{r_{st}} = r_{st}^3 \rho_s \rho_t d\rho_s d\rho_t d\varphi
\end{aligned}
$$

[1] Suggested by Saul Basri.

$$
\begin{aligned}
& \mathbf{G}_{ts\ell 1}(\text{part1}) \\
= {} & \frac{k}{2}\int_0^1 \int_{1-\rho_s}^{1+\rho_s} \frac{\mathbf{v}_t r_{st}^2 \left(\rho_s^2+\rho_t^2-1\right)}{r_{st}^3 \rho_s^3 r_{st}^3 \rho_t^3} r_{st}^3 \rho_s \rho_t d\rho_s d\rho_t d\varphi \\
= {} & \frac{\pi k \mathbf{v}_t}{r_{st}} \int_0^1 \int_{1-\rho_s}^{1+\rho_s} \frac{\rho_s^2+\rho_t^2-1}{\rho_s^2 \rho_t^2} d\rho_s d\rho_t \\
= {} & \frac{\pi k \mathbf{v}_t}{r_{st}} \int_0^1 \int_{1-\rho_s}^{1+\rho_s} \left(\frac{1}{\rho_t^2}+\frac{1}{\rho_s^2}-\frac{1}{\rho_s^2\rho_t^2}\right) d\rho_s d\rho_t \qquad (9.24)\\
= {} & \frac{\pi k \mathbf{v}_t}{r_{st}} \int_0^1 \left(\frac{2\rho_s}{1-\rho_s^2}+\frac{2\rho_s}{\rho_s^2}-\frac{1}{\rho_s^2}\frac{2\rho_s}{\left(1-\rho_s^2\right)}\right) d\rho_s \\
= {} & \frac{\pi k \mathbf{v}_t}{r_{st}} \int_0^1 \frac{2\rho_s^3+2\rho_s\left(1-\rho_s^2\right)-2\rho_s}{\left(1-\rho_s^2\right)} d\rho_s = \left(\frac{\pi k \mathbf{v}_t}{r_{st}}\right) 0 = 0 \qquad (9.25)
\end{aligned}
$$

$$
\begin{aligned}
& \mathbf{G}_{ts\ell 1}\,(\text{part 2}) \\
= {} & \frac{\pi k \mathbf{v}_t}{r_{st}} \int_1^\infty \int_{\rho_s-1}^{\rho_s+1} \left(\frac{1}{\rho_t^2}+\frac{1}{\rho_s^2}-\frac{1}{\rho_s^2\rho_t^2}\right) d\rho_s d\rho_t \\
= {} & \frac{\pi k \mathbf{v}_t}{r_{st}} \int_1^\infty \left(\frac{2}{\rho_s^2-1}+\frac{2}{\rho_s^2}-\frac{1}{\rho_s^2}\frac{2}{\left(\rho_s^2-1\right)}\right) d\rho_s = \frac{\pi k \mathbf{v}_t}{r_{st}} \int_1^\infty \frac{2\rho_s^2+2\left(\rho_s^2-1\right)-2}{\rho_s^2\left(\rho_s^2-1\right)} d\rho_s \\
= {} & \frac{\pi k \mathbf{v}_t}{r_{st}} \int_1^\infty \frac{4\left(\rho_s^2-1\right)-2}{\rho_s^2\left(\rho_s^2-1\right)} d\rho_s = \frac{4\pi k \mathbf{v}_t}{r_{st}} \int_1^\infty \frac{d\rho_s}{\rho_s^2} = \frac{4\pi k \mathbf{v}_t}{r_{st}} \left[-\frac{1}{\rho_s^2}\right]_1^\infty = \frac{4\pi k}{3 r_{st}} \left(3\mathbf{v}_t\right)
\end{aligned}
$$

Thus

$$
\begin{aligned}
\mathbf{G}_{ts\ell 1} &= \int\int \mathbf{g}_{ts\ell 1} d\tau = \frac{4\pi k \mathbf{v}_t}{r_{st}} = \frac{4\pi}{\left(4\pi\right)^2 \left(\varepsilon_0 c^2\right)} \frac{\mathbf{v}_t}{r_{st}} \\
\mathbf{G}_{ts\ell 1} &= \frac{q_t q_s}{4\pi\varepsilon_0 c^2} \frac{\mathbf{v}_t}{r_{st}} \qquad (9.26)
\end{aligned}
$$

Thus Eq. (9.26) agrees with Eq. (9.23).

Now evaluate the integral of the second term in Eq. (9.17)., call it $\mathbf{g}_{ts\ell 2}$.

$$
\mathbf{g}_{ts\ell 2} = k \frac{-\mathbf{r}_t \left(\mathbf{v}_t \cdot \mathbf{r}_s\right)}{r_s^3 r_t^3}
$$

Use $\mathbf{r}_t = \mathbf{r}_s - \mathbf{r}_{st}$.

$$\begin{aligned}
\mathbf{g}_{ts\ell 2} &= -\frac{k\mathbf{r}_t\left(\mathbf{v}_t\cdot\mathbf{r}_s\right)}{r_s^3 r_t^3}\frac{r_s r_t}{r_{st}}dr_s dr_t d\varphi \qquad k = q_s q_t/\left(4\pi\right)^2\varepsilon_0 c^2\\
&= \frac{k}{r_{st}}\frac{\left[-\mathbf{r}_s\left(\mathbf{v}_t\cdot\mathbf{r}_s\right)+\mathbf{r}_{st}\left(\mathbf{v}_t\cdot\mathbf{r}_s\right)\right]}{r_s^2 r_t^2}dr_s dr_t d\varphi\\
\mathbf{g}_{ts\ell 2} &= \frac{-k}{r_{st}}\frac{\mathbf{r}_s\left(\mathbf{v}_t\cdot\mathbf{r}_s\right)}{r_s^2 r_t^2}dr_s dr_t d\varphi + \frac{k}{r_{st}}\mathbf{r}_s\left(\mathbf{v}_t\cdot\mathbf{r}_s\right)\frac{dr_s dr_t d\varphi}{r_s^2 r_t^2} \qquad (9.27)
\end{aligned}$$

Let

$$\begin{aligned}
\mathbf{v}_t &= v_{tx}\mathbf{e}_1 + v_{ty}\mathbf{e}_2 + v_{tz}\mathbf{e}_3\\
\mathbf{r}_s &= r_s\left[\mathbf{e}_1\sin\theta\cos\varphi + \mathbf{e}_2\sin\theta\sin\varphi + \mathbf{e}_3\cos\theta\right]\\
\mathbf{v}_t\cdot\mathbf{r}_s &= r_s\left[v_{tx}\sin\theta\cos\varphi + v_{ty}\sin\theta\sin\varphi + v_{tz}\cos\theta\right]
\end{aligned}$$

$$\begin{aligned}
\mathbf{r}_s\left(\mathbf{v}_t\cdot\mathbf{r}_s\right) &= r_s^2\left\{\mathbf{e}_1\left[\underline{v_{tx}\sin^2\theta\cos^2\varphi} + v_{ty}\sin^2\theta\cos\varphi\sin\varphi + v_{tz}\sin\theta\cos\theta\cos\varphi\right]\right.\\
&+\mathbf{e}_2\left[v_{tx}\sin^2\theta\sin\varphi\cos\varphi + \underline{v_{ty}\sin^2\theta\sin^2\varphi} + v_{tz}\sin\theta\cos\theta\sin\varphi\right]\\
&\left.+\mathbf{e}_3\left[v_{tx}\sin\theta\cos\theta\cos\varphi + v_{ty}\sin\theta\cos\theta\sin\varphi + \underline{v_{tz}\cos^2\theta}\right]\right\} \qquad (9.28)
\end{aligned}$$

To integrate Eq. (9.28) over φ, use

$$\begin{aligned}
\int_0^{2\pi}\cos\varphi d\varphi &= 0, \qquad \int_0^{2\pi}\sin\varphi d\varphi = 0, \qquad \int_0^{2\pi}\sin\varphi\cos\varphi d\varphi = 0\\
\int_0^{2\pi}\cos^2\varphi d\varphi &= \pi, \qquad \int_0^{2\pi}\sin^2\varphi d\varphi = \pi \qquad \int_0^{2\pi}d\varphi = 2\pi
\end{aligned}$$

After integrating over φ, the integrand of the first sum in Eq. (9.27) is

$$-\frac{k}{r_{st}}\int_0^{2\pi}\mathbf{r}_s\left(\mathbf{r}_s\cdot\mathbf{v}_t\right)d\varphi = -\frac{k}{r_{st}}r_s^2\pi\left[\left(\mathbf{e}_1 v_{tx} + \mathbf{e}_2 v_{ty}\right)\sin^2\theta + \mathbf{e}_3 2v_{tz}\cos^2\theta\right]dr_s dr_t/r_s^2 r_t^2$$

Thus, Eq. (9.27), after integrating over φ, becomes

$$\begin{aligned}
\int\mathbf{g}_{ts\ell 2}d\tau &= \frac{k\pi}{r_{st}}\left(\int\int\left(-r_s^2\left(\mathbf{e}_1 v_{tx} + \mathbf{e}_2 v_{ty}\right)\sin^2\theta - \mathbf{e}_3 2r_s^2 v_{tz}\cos^2\theta\right.\right.\\
&\left.\left.+\mathbf{e}_3 2r_{st} r_s v_{tz}\cos\theta\right)\frac{dr_s dr_t}{r_s^2 r_t^2}\right) \qquad (9.29)
\end{aligned}$$

Add to Eq. (9.29), $\frac{k\pi}{r_{st}}\left(r_s^2 v_{tz}\mathbf{e}_3 - r_s^2 v_{tz}\mathbf{e}_3\right)\sin^2\theta$, which is identically 0, to obtain

$$\begin{aligned}\int \mathbf{g}_{ts\ell 2}d\tau &= \frac{k\pi}{r_{st}}\int\int\left[-r_s^2\left(\mathbf{e}_1 v_{tx}+\mathbf{e}_2 v_{ty}+\mathbf{e}_3 v_{tz}\right)\sin^2\theta - \mathbf{e}_3 r_s^2 v_{tz}\sin^2\theta\right.\\ &\quad \left.-\mathbf{e}_3 2r_s^2 v_{tz}\cos^2\theta + \mathbf{e}_3 2r_{st}r_s v_{tz}\cos\theta\right]\frac{dr_s dr_t}{r_s^2 r_t^2}\\ \int \mathbf{g}_{ts\ell 2}d\tau &= \frac{k\pi}{r_{st}}\int\int\left[r_s^2\left(-\mathbf{v}_t + v_{tz}\mathbf{e}_3\right)\sin^2\theta\right.\\ &\quad \left.-2\mathbf{e}_3 v_{tz} r_s^2\cos^2\theta + 2\mathbf{e}_3 r_{st}r_s v_{tz}\cos\theta\right]\frac{dr_s dr_t}{r_s^2 r_t^2} \qquad (9.30)\end{aligned}$$

To evaluate the integral of Eq. (9.30), use the dimensionless variables.

$$\rho_s = \frac{r_s}{r_{st}}, \qquad \rho_t = \frac{r_t}{r_{st}}, \qquad d\tau = \frac{r_s r_t dr_s dr_t}{r_{st}} = r_{st}^3\rho_s\rho_t d\rho_s d\rho_t$$

and integrate all three terms as a unit. Integrated individually, each of the cosine terms are singular. Therefore we lump the cosine terms $\cos^2\theta$ and $\cos\theta$ together. $\sin^2\theta$ integrates okay by itself. Nevertheless, in places we add the integrands of all three.

Use

$$\begin{aligned} r_t^2 &= r_s^2 + r_{st}^2 - 2r_{st}r_s\cos\theta\\ \rho_t^2 &= \rho_s^2 + 1 - 2\rho_s\cos\theta\\ 2\rho_s\cos\theta &= 1+\rho_s^2-\rho_t^2\\ \cos\theta &= \left[\left(1+\rho_s^2\right)-\rho_t^2\right]/2\rho_s\\ \cos^2\theta &= \left[\left(1+\rho_s^2\right)^2-\rho_t^2\right]^2/4\rho_s^2 \qquad (9.31)\\ &= \left[\left(1+\rho_s^2\right)^2 - 2\left(1+\rho_s^2\right)\rho_t^2+\rho_t^4\right]/4\rho_s^2\end{aligned}$$

$$\begin{aligned}\sin^2\theta &= 1-\cos^2\theta = 1-\frac{\left(1+\rho_s^2\right)^2-2\left(1+\rho_s^2\right)\rho_t^2+\rho_t^4}{4\rho_s^2}\\ \sin^2\theta &= \frac{4\rho_s^2-\left(1+\rho_s^2\right)^2+2\left(1+\rho_s^2\right)\rho_t^2-\rho_t^4}{4\rho_s^2}\\ &= \left[2\rho_s^2-1-\rho_s^4+2\left(1+\rho_s^2\right)\rho_t^2-\rho_t^4\right]/4\rho_s^2\\ \sin^2\theta &= -\left[\left(1-\rho_s^2\right)^2-2\left(1+\rho_s^2\right)\rho_t^2+\rho_t^4\right]/4\rho_s^2 \qquad (9.32)\end{aligned}$$

Thus,

$$\int \mathbf{g}_{ts\ell 2} d\tau = \frac{k\pi}{r_{st}} \Bigg\{ r_s^2 \rho_s^2 \left(-\mathbf{v}_t + v_{tz}\mathbf{e}_3\right) \int\int \overbrace{\frac{-\left[\left(1-\rho_s^2\right)^2 - 2\left(1+\rho_s^2\right)\rho_t^2 + \rho_t^4\right]}{4\rho_s^2}}^{\sin^2\theta}$$
$$-2\mathbf{e}_3 v_{tz} r_s^2 \rho_s^2 \overbrace{\frac{\left[\left(1+\rho_s^2\right) - \rho_t^2\right]^2}{4\rho_s^2}}^{\cos^2\theta}$$
$$+2\mathbf{e}_3 r_{st}^2 \rho_s v_{tz} \overbrace{\frac{\left[\left(1+\rho_s^2\right) - \rho_t^2\right]}{2\rho_s}}^{\cos\theta} \Bigg\} \frac{d\rho_s d\rho_t}{r_{st}^2 \rho_s^2 \rho_t^2} \tag{9.33}$$

Thus the integrals to be evaluated are:

$$\int \mathbf{g}_{ts\ell 2} d\tau =$$

$$\frac{k\pi}{r_{st}} \Bigg[\left(\mathbf{v}_t - \mathbf{e}_3 v_{tz}\right) \int_0^1 \int_{1-\rho_s}^{1+\rho_s} \overbrace{\frac{\left[\left(1-\rho_s^2\right)^2 - 2\left(1+\rho_s^2\right)\rho_t^2 + \rho_t^4\right]}{4\rho_s^2\rho_t^2}}^{\sin^2\theta} d\rho_t d\rho_s \tag{9.34}$$

$$-2\mathbf{e}_3 v_{tz} \int_0^1 \int_{1-\rho_s}^{1+\rho_s} \overbrace{\frac{\left[\left(1+\rho_s^2\right)^2 - 2\left(1+\rho_s^2\right)\rho_t^2 + \rho_t^4\right]}{4\rho_s^2\rho_t^2}}^{\cos^2\theta} d\rho_t d\rho_s \tag{9.35}$$

$$+2\mathbf{e}_3 v_{tz} \int_0^1 \int_{1-\rho_s}^{1+\rho_s} \overbrace{\frac{\left[\left(1+\rho_s^2\right) - \rho_t^2\right]}{2\rho_s^2\rho_t^2}}^{\cos\theta} d\rho_t d\rho_s \Bigg] \tag{9.36}$$

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$$+ \quad \frac{k\pi}{r_{st}}\left[(\mathbf{v}_t - \mathbf{e}_3 v_{tz})\int_1^\infty\int_{\rho_s-1}^{\rho_s+1}\frac{\overbrace{\left[\left(1-\rho_s^2\right)^2 - 2\left(1+\rho_s^2\right)\rho_t^2 + \rho_t^4\right]}^{\sin^2\theta}}{4\rho_s^2\rho_t^2}d\rho_t d\rho_s \right. \tag{9.37}$$

$$-2\mathbf{e}_3 v_{tz}\int_1^\infty\int_{\rho_s-1}^{\rho_s+1}\frac{\overbrace{\left[\left(1+\rho_s^2\right)^2 2 - 2\left(1+\rho_s^2\right)\rho_t^2 + \rho_t^4\right]}^{\cos^2\theta}}{4\rho_s^2\rho_t^2}d\rho_t d\rho_s \tag{9.38}$$

$$\left. +2\mathbf{e}_3 v_{tz}\int_1^\infty\int_{\rho_s-1}^{\rho_s+1}\frac{\overbrace{\left[\left(1+\rho_s^2\right) - \rho_t^2\right]}^{\cos\theta}}{2\rho_s^2\rho_t^2}d\rho_t d\rho_s\right] \tag{9.39}$$

The following integrals appear in the above.

$$\int_0^1 \quad \begin{array}{ll} \displaystyle\int_{1-\rho_s}^{1+\rho_s}\frac{d\rho_t}{\rho_t^2} = \left[-\frac{1}{\rho_t}\right] = \frac{2\rho_s}{1-\rho_s^2}, & \displaystyle\int_{\rho_s-1}^{\rho_s+1}\frac{d\rho_t}{\rho_t^2} = \frac{2}{\rho_s^2-1} \\ \displaystyle\int_{1-\rho_s}^{1+\rho_s} d\rho_t = [\rho_t] = 2\rho_s, & \displaystyle\int_{\rho_s-1}^{\rho_s+1} d\rho_t = [\rho_t] = 2 \\ \displaystyle\int_{1-\rho_s}^{1+\rho_s}\rho_t^2 d\rho_t = \frac{1}{3}\left[\rho_t^3\right] = 2\rho_s\left(1+\frac{\rho_s^2}{3}\right), & \displaystyle\int_{\rho_s-1}^{\rho_s+1}\rho_t^2 d\rho_t = \frac{1}{3}\left[\rho_t^3\right] = 2\left(\rho_s^2+\frac{1}{3}\right) \end{array} \quad \int_1^\infty$$

Substituting the results of the ρ_t integration, we obtain from Eqs. (9.34-9.36)

$$\int_0^1 \mathbf{g}_{ts\ell 2} d\tau =$$

$$\frac{k\pi}{r_{st}}\left((\mathbf{v}_t - \mathbf{e}_3 v_{tz})\int_0^1\frac{1}{4\rho_s^2}\overbrace{\left[\frac{\left(1-\rho_s^2\right)^2 2\rho_s}{1-\rho_s^2} - 2\left(1+\rho_s^2\right)2\rho_s + 2\rho_s\left(1+\frac{\rho_s^2}{3}\right)\right]}^{-\sin^2\theta}d\rho_s \right. \tag{9.40}$$

$$\left. -2\mathbf{e}_3 v_{tz}\int_0^1\frac{1}{4\rho_s^2}\overbrace{\left[\frac{\left(1+\rho_s^2\right)^2 2\rho_s}{\left(1-\rho_s^2\right)} - 2\left(1+\rho_s^2\right)2\rho_s + 2\rho_s\left(1+\frac{\rho_s^2}{3}\right)\right]}^{\cos^2\theta}d\rho_s\right) \tag{9.41}$$

Let $\frac{k\pi 2\mathbf{e}_3 v_{tz}}{r_{st}} = A$, then

$$\begin{aligned}
&\frac{k\pi 2\mathbf{e}_3 v_{tz}}{r_{st}} \int_0^1 \int_{1-\rho_s}^{1+\rho_s} \frac{\overset{\cos\theta}{[1+\rho_s^2] - \rho_t^2}}{2\rho_s^2\rho_t^2} d\rho_t d\rho_s \\
= \;& A \int_0^1 \left[\frac{\left(1+\rho_s^2\right) \frac{2\rho_s}{(1-\rho_s^2)}}{2\rho_s^2} - \frac{2\rho_s}{\rho_s^2} \right] d\rho_s = A \int_0^1 \left[\frac{\left(1+\rho_s^2\right) 2\rho_s - 2\rho_s\left(1-\rho_s^2\right)}{2\left(1-\rho_s^2\right)\rho_s^2} \right] d\rho_s \\
= \;& A \int_0^1 \frac{1}{\rho_s} \left[\frac{\left(1+\rho_s^2\right) - \left(1-\rho_s^2\right)}{\left(1-\rho_s^2\right)} \right] d\rho_s = A \int_0^1 \frac{1}{\rho_s} \left[\frac{2\rho_s^2}{\left(1-\rho_s^2\right)} \right] d\rho_s \\
= \;& A \int_0^1 \frac{\overset{\cos\theta}{2\rho_s}}{\left(1-\rho_s^2\right)} d\rho_s \qquad (9.42)
\end{aligned}$$

Substituting the results of the ρ_t integration, we obtain from Eqs. (9.37-9.39)

$$\int_1^\infty \mathbf{g}_{ts\ell 2} d\tau =$$

$$\frac{k\pi}{r_{st}} \left(\mathbf{v}_t - \mathbf{e}_3 v_{tz}\right) \int_1^\infty \frac{1}{4\rho_s^2} \left[\frac{\left(1-\rho_s^2\right)^2 2}{\rho_s^2 - 1} \overset{-\sin^2\theta}{-\, 2\left(1+\rho_s^2\right) 2 + 2\left(\frac{3\rho_s^2+1}{3}\right)} \right] d\rho_s \qquad (9.43)$$

$$-\frac{k\pi}{r_{st}} \frac{2\mathbf{e}_3 v_{tz}}{4} \int_1^\infty \frac{1}{\rho_s^2} \left[\frac{\left(1+\rho_s^2\right)^2 2}{\rho_s^2 - 1} \overset{\cos^2\theta}{-\, 2\left(1+\rho_s^2\right) 2 + 2\left(\rho_s^2 + \frac{1}{3}\right)} \right] d\rho_s \qquad (9.44)$$

$$\begin{aligned}
&\frac{k\pi}{r_{st}} 2\mathbf{e}_3 v_{tz} \int_1^\infty \int_{\rho_s - 1}^{\rho_s + 1} \frac{\overset{\cos\theta}{\left(1+\rho_s^2\right) - \rho_t^2}}{2\rho_s^2\rho_t^2} d\rho_t d\rho_s \;=\; \frac{k\pi}{r_{st}} 2\mathbf{e}_3 v_{tz} \int_1^\infty \frac{\left(1+\rho_s^2\right) \dfrac{2}{\rho_s^2 - 1} - 2}{2\rho_s^2} d\rho_s \\
&= \frac{k\pi}{r_{st}} 2\mathbf{e}_3 v_{tz} \int_1^\infty \frac{\left(1+\rho_s^2\right) 2 - 2\left(\rho_s^2 - 1\right)}{2\rho_s^2\left(\rho_s^2 - 1\right)} d\rho_s = \frac{k\pi}{r_{st}} 2\mathbf{e}_3 v_{tz} \int_1^\infty \frac{1 + \rho_s^2 - \rho_s^2 + 1}{\rho_s^2\left(\rho_s^2 - 1\right)} d\rho_s \\
&= \frac{k\pi}{r_{st}} 2\mathbf{e}_3 v_{tz} \int_1^\infty \frac{\overset{\cos\theta}{2}}{\rho_s^2\left(\rho_s^2 - 1\right)} d\rho_s \qquad (9.45)
\end{aligned}$$

Consider Eq. (9.40)

$$\begin{aligned}
&\frac{\pi k}{r_{st}}\int_0^1 g_{ts\ell 2}d\tau \\
=\ & \frac{\pi k}{r_{st}}\left[(\mathbf{v}_t-\mathbf{e}_3 v_{tz})\int_0^1 \frac{1}{4}\left(\frac{2\rho_s}{\rho_s^2}\right)\left[\overset{-\sin^2\theta}{\left(1-\rho_s^2\right)-2\left(1+\rho_s^2\right)+\left(1+\frac{\rho_s^2}{3}\right)}\right]d\rho_s\right] \\
=\ & \frac{\pi k}{r_{st}}\left[(\mathbf{v}_t-\mathbf{e}_3 v_{tz})\int_0^1 \frac{1}{4}\left(\frac{2}{\rho_s}\right)\left[\frac{(3-3\rho_s^2-6-6\rho_s^2+3+\rho_s^2)}{3}\right]d\rho_s\right] \\
=\ & \frac{\pi k}{r_{st}}\left[(\mathbf{v}_t-\mathbf{e}_3 v_{tz})\int_0^1 \frac{1}{2\rho_s}\frac{[-8\rho_s^2]}{3}d\rho_s\right]=\frac{\pi k}{r}\left[(\mathbf{v}_t-\mathbf{e}_3 v_{tz})\left[\frac{-4}{3}\frac{\rho_s^2}{2}\right]_0^1\right] \qquad (9.46)
\end{aligned}$$

$$\begin{aligned}
\int_0^1 g_{ts\ell 2}d\tau &= -\left(\frac{k\pi}{r_{st}}\right)\frac{2}{3}\left(\mathbf{v}_t-\mathbf{e}_3 v_{tz}\right) \\
&= -\left(\frac{k\pi}{r_{st}}\right)\frac{2}{3}\mathbf{v}_t+\left(\frac{k\pi}{r_{st}}\right)\frac{2}{3}\mathbf{e}_3 v_{tz} \\
\text{Terms involving } -\sin^2\theta \text{ are } &= \frac{4k\pi}{3r_{st}}\left(-\frac{\mathbf{v}_t}{2}+\frac{\mathbf{e}_3 v_{tz}}{2}\right) \qquad (9.47)
\end{aligned}$$

Consider Eq. (9.41) ($\cos^2\theta$)

$$\begin{aligned}
&\int_0^1 g_{ts\ell 2}d\tau = \\
&\frac{k\pi}{r}\left[-2\mathbf{e}_3 v_{tz}\int_0^1 \frac{1}{4\rho_s^2}\left(\overset{\cos^2\theta}{\frac{\left(1+\rho_s^2\right)^2 2\rho_s}{1-\rho_s^2}-2\left(1+\rho_s^2\right)2\rho_s+2\rho_s\left(1+\frac{\rho_s^2}{3}\right)}\right)\right]d\rho_s \\
=\ & \frac{k\pi}{r_{st}}2\mathbf{e}_3 v_{tz}\int_0^1 -\frac{1}{2}\left(\frac{1}{\rho_s}\right)\times \\
&\left(\overset{\cos^2\theta}{\frac{3\left(1+\rho_s^2\right)^2-6\left(1+\rho_s^2\right)\left(1-\rho_s^2\right)+\left(1-\rho_s^2\right)\left(3+\rho_s^2\right)}{3\left(1-\rho_s^2\right)}}\right)d\rho_s
\end{aligned}$$

$$= \frac{k\pi}{r_{st}}\left[-2\mathbf{e}_3 v_{tz}\int_0^1 \frac{1}{2}\frac{1}{\rho_s}\left(\frac{\overset{\cos^2\theta}{3+6\rho_s^2+3\rho_s^4-6+6\rho_s^4+3-2\rho_s^2-\rho_s^4}}{3\left(1-\rho_s^2\right)}\right)\right]d\rho_s$$

$$= -\frac{k\pi}{r_{st}}2\mathbf{e}_3 v_{tz}\int_0^1 \frac{1}{2}\frac{1}{\rho_s}\left(\frac{4\rho_s^2+8\rho_s^4}{3\left(1-\rho_s^2\right)}\right)d\rho_s$$

$$\cos^2\theta \text{ terms are} = -\frac{k\pi}{r_{st}}2\mathbf{e}_3 v_{tz}\int_0^1 \frac{1}{2}\left(\frac{4\rho_s}{3}\right)\frac{\left(1+2\rho_s^2\right)}{1-\rho_s^2}d\rho_s \tag{9.48}$$

Consider Eq. (9.45) ($\cos\theta$ term)

$$\cos\theta \text{ terms are } = \frac{k\pi}{r}2\mathbf{e}_3 v_{tz}\int_0^1 \frac{2\rho_s}{1-\rho_s^2}dp_s \tag{9.49}$$

Add Eq. (9.46) $\left(\sin^2\theta\right)$ to Eqs. (9.48) + (9.49). Eq. (9.47) is:

$$\int_0^1 g_{ts\ell 2}d\tau = \frac{4k\pi}{3r_{st}}\left[\overset{-\sin^2\theta}{\frac{-\mathbf{v}_t}{2}+\frac{\mathbf{e}_3 v_{tz}}{2}}\right]$$

Eqs. (9.48) and (9.49) are:

$$= \frac{k\pi}{r_{st}}2\mathbf{e}_3 v_{tz}\left(\int_0^1 \overset{\cos^2\theta\ (9.48)}{-\frac{2\rho_s\left(1+2\rho_s^2\right)}{3\left(1-\rho_s^2\right)}}+\overset{\cos\theta\ \ (9.49)}{\frac{2\rho_s}{\left(1-\rho_s^2\right)}d\rho_s}\right)$$

$$= \frac{k\pi}{r_{st}}2\mathbf{e}_3 v_{tz}\left[\int_0^1 \frac{-2\rho_s\left(1+2\rho_s^2\right)+6\rho_s}{3\left(1-\rho_s^2\right)}\right]d\rho_s = \frac{k\pi}{r_{st}}2\mathbf{e}_3 v_{tz}\int_0^1\left[\frac{\left(4\rho_s-4\rho_s^3\right)}{3\left(1-\rho_s^2\right)}\right]d\rho_s$$

$$= \frac{k\pi}{r_{st}}2\mathbf{e}_3 v_{tz}\left[\int_0^1 \frac{4\rho_s\left(1-\rho_s^2\right)}{3\left(1-\rho_s^2\right)}d\rho_s\right] = \frac{k\pi}{r_{st}}2\mathbf{e}_3 v_{tz}\int_0^1 \frac{4}{3}\rho_s d\rho_s = \frac{k\pi}{r_{st}}2\mathbf{e}_3 v_{tz}\frac{2}{3}$$

Thus Eq. (9.48b) + Eq. (9.49) is:

$$\int_0^1 g_{ts\ell 2}d\tau = \overset{\cos^2\theta+\cos\theta}{\frac{4k\pi}{3r_{st}}\mathbf{e}_3 v_{tz}}$$

In toto: the $\int_0^1$ integral has the value

$$\int_0^1 g_{ts\ell 2}d\tau = \frac{4k\pi}{3r_{st}}\left[\overset{-\sin^2\theta}{-\frac{\mathbf{v}_t}{2}+\frac{\mathbf{e}_3 v_{tz}}{2}}+\overset{\cos^2\theta+\cos\theta}{\mathbf{e}_3 v_{tz}}\right]$$

$$\int_0^1 g_{ts\ell 2}d\tau = \frac{4k\pi}{3r_{st}}\left[\overset{-\sin^2\theta+\cos^2\theta+\cos\theta}{-\frac{\mathbf{v}_t}{2}+\frac{3}{2}\mathbf{e}_3 v_{tz}}\right] \tag{9.50}$$

Next add Eqs. (9.43-9.45) to obtain

$$
\begin{aligned}
&\int_1^\infty g_{ts\ell 2} \\
= \;& \frac{k\pi}{r_{st}} \left(\frac{(\mathbf{v}_t - \mathbf{e}_3 v_{tz})}{2} \int_1^\infty \frac{1}{\rho_s^2} \left[\overbrace{-\left(1-\rho_s^2\right) - 2\left(1+\rho_s^2\right) + \left(\rho_s^2 + \frac{1}{3}\right)}^{-\sin^2\theta} \right] d\rho_s \right. \\
& -2\mathbf{e}_3 v_{tz} \int_1^\infty \frac{1}{2\rho_s^2} \left[\overbrace{\frac{\left(1+\rho_s^2\right)^2}{\left(\rho_s^2 - 1\right)} - 2\left(1+\rho_s^2\right) + \frac{3\rho_s^2+1}{3}}^{\cos^2\theta} \right] d\rho_s \\
& \left. +2\mathbf{e}_3 v_{tz} \int_1^\infty \left[\frac{2}{\rho_s^2} \overbrace{\frac{1}{\left(\rho_s^2-1\right)}}^{\cos\theta} \right] d\rho_s \right) \qquad (9.51)
\end{aligned}
$$

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$$
\begin{aligned}
= \;& \frac{k\pi}{r_{st}} \left(\frac{(\mathbf{v}_t - \mathbf{e}_3 v_{tz})}{2} \int_1^\infty \frac{1}{\rho_s^2} \left[\overbrace{\frac{-3\left(1-\rho_s^2\right) - 6\left(1+\rho_s^2\right) + 3\rho_s^2 + 1}{3}}^{-\sin^2\theta} \right] d\rho_s \right. \\
& -2\mathbf{e}_3 v_{tz} \int_1^\infty \frac{1}{2\rho_s^2} \left[\overbrace{\frac{3\left(1+\rho_s^2\right)^2 - 6\left(\rho_s^2-1\right)\left(1+\rho_s^2\right) + \left(\rho_s^2-1\right)\left(3\rho_s^2+1\right)}{3\left(\rho_s^2-1\right)}}^{\cos^2\theta} \right] d\rho_s \\
& +2\mathbf{e}_3 v_{tz} \int_1^\infty \left[\frac{2}{\rho_s^2} \overbrace{\frac{1}{\left(\rho_s^2-1\right)}}^{\cos\theta} \right] d\rho_s \qquad (9.52)
\end{aligned}
$$

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$$
\begin{aligned}
= \;& \frac{k\pi}{r_{st}} \left((\mathbf{v}_t - \mathbf{e}_3 v_{tz}) \int_1^\infty \frac{1}{2\rho_s^2} \left[\overbrace{\frac{-3 + 3\rho_s^2 - 6 - 6\rho_s^2 + 3\rho_s^2 + 1}{3}}^{-\sin^2\theta} \right] d\rho_s \right. \\
& -2\mathbf{e}_3 v_{tz} \int_1^\infty \frac{1}{2\rho_s^2} \left[\overbrace{\frac{3 + 6\rho_s^2 + 3\rho_s^4 + 6 - 6\rho_s^4 + 3\rho_s^4 + \rho_s^2 - 3\rho^2 - 1}{3\left(\rho_s^2-1\right)}}^{\cos^2\theta} \right] d\rho_s \\
& \left. +2\mathbf{e}_3 v_{tz} \int_1^\infty \frac{2}{\rho_s^2} \left[\overbrace{\frac{1}{\left(\rho_s^2-1\right)}}^{\cos\theta} \right] d\rho_s \right) \qquad (9.53)
\end{aligned}
$$

Add the $cos^2\theta$ and $\cos\theta$ terms in Eq. (9.53)

$$\begin{aligned}
&\int_1^\infty g_{ts\ell 2} d\tau \\
&= \frac{k\pi}{r_{st}} \left(-2\mathbf{e}_3 v_{tz} \int_1^\infty \left[\frac{1}{2\rho_s^2} \overbrace{\left(\frac{8+4\rho_s^2}{3\left(\rho_s^2-1\right)} \right)}^{\cos^2\theta} - \frac{2}{\rho_s^2} \overbrace{\frac{1}{\left(\rho_s^2-1\right)}}^{\cos\theta} \right] d\rho_s \right) \\
&= \frac{k\pi}{r_{st}} \left(-2\mathbf{e}_3 v_{tz} \int_1^\infty \frac{1}{\rho_s^2} \left[\left(\frac{4+2\rho_s^2}{3\left(\rho_s^2-1\right)} \right) - \frac{6}{3\left(\rho_s^2-1\right)} \right] d\rho_s \right) \\
&= \frac{k\pi}{r_{st}} \left(-2\mathbf{e}_3 v_{tz} \right) \int_1^\infty \frac{1}{\rho_s^2} \left[\frac{-2+2\rho_s^2}{3\left(\rho_s^2-1\right)} \right] d\rho_s \\
&= \frac{k\pi}{r_{st}} \left(-2\mathbf{e}_3 v_{tz} \right) \int_1^\infty \frac{2\left(\rho_s^2-1\right)}{\rho_s^2 3\left(\rho_s^2-1\right)} d\rho_s \\
&= \frac{k\pi}{r_{st}} \left(-2\mathbf{e}_3 v_{tz} \right) \frac{2}{3} \int_1^\infty \frac{d\rho_s}{\rho_s^2} = -\frac{4}{3} \frac{k\pi}{r_{st}} \mathbf{e}_3 v_{tz} \left[-\frac{1}{\rho_s} \right]_1^\infty = -\frac{4k\pi}{3r_{st}} \mathbf{e}_3 v_{tz}
\end{aligned}$$

Therefore

$$\int_1^\infty g_{ts\ell 2} d\tau \left(\cos^2\theta + \cos\theta \right) = -\frac{4}{3} \frac{k\pi}{r_{st}} \mathbf{e}_3 v_{tz} \tag{9.54}$$

The $-\sin^2\theta$ term in Eq. (9.53) is

$$\begin{aligned}
\frac{k\pi}{r} \left(\mathbf{v}_t - \mathbf{e}_3 v_{tz} \right) \int_1^\infty -\frac{4}{3} \frac{d\rho_s}{\rho_s^2} &= \frac{k\pi}{r} \left(\mathbf{v}_t - \mathbf{e}_3 v_{tz} \right) \left(-\frac{4}{3} \left[-\frac{1}{\rho_s} \right]_1^\infty \right) \\
&= \frac{4}{3} \frac{k\pi}{r} \left(-\mathbf{v}_t + \mathbf{e}_3 v_{tz} \right)
\end{aligned} \tag{9.55}$$

$$\int_1^\infty g_{ts\ell 2} d\tau = \frac{4k\pi}{3} \left(\overbrace{-\mathbf{v}_t + \mathbf{e}_3 v_{tz}}^{-\sin^2\theta} - \overbrace{\mathbf{e}_3 v_{tz}}^{\cos^2\theta + \cos\theta} \right)$$

By Eq. (9.23),

$$\mathbf{G}_{ts\ell 1} = \frac{4k\pi\mathbf{v}_t}{r_{st}} = \frac{4k\pi}{3}\left(3\mathbf{v}_t\right) \tag{9.56}$$

$$\int_0^1 g_{ts\ell 2} d\tau = \frac{4k\pi}{3r_{st}}\left(-\frac{\mathbf{v}_t}{2} + \frac{3}{2}\mathbf{e}_3 v_{tz}\right) \tag{9.57}$$

$$\int_1^\infty g_{ts\ell 2} d\tau + \int_0^1 g_{ts\ell 2} d\tau = \frac{4k\pi}{3}\left(-\mathbf{v}_t - \frac{\mathbf{v}_t}{2} + \frac{3}{2}\mathbf{e}_3 v_{tz}\right)$$

$$\begin{aligned}
\mathbf{G}_{ts\ell 1} + \int_1^\infty g_{ts\ell 2} d\tau + \int_0^1 g_{ts\ell 2} d\tau &= \frac{4k\pi}{3r_{st}}\left(3\mathbf{v}_t - \mathbf{v}_t - \frac{\mathbf{v}_t}{2} + \frac{3}{2}\mathbf{e}_3 v_{tz}\right) \\
&= \frac{4k\pi}{3r_{st}}\left(\frac{3}{2}\mathbf{v}_t + \frac{3}{2}\mathbf{e}_3 v_{tz}\right) \\
&= \frac{4k\pi}{r_{st}}\frac{1}{2}\left(\mathbf{v}_t + \mathbf{e}_3 v_{tz}\right)
\end{aligned} \tag{9.58}$$

$$k = \frac{q_s q_t}{(4\pi)^2 \varepsilon_0 c^2} \qquad \frac{4\pi}{r_{st}}\frac{q_s q_t}{(4\pi)^2}\varepsilon_0 c^2 = \frac{q_s q_t}{4\pi\varepsilon_0 c^2 r_{st}} \qquad \frac{\mathbf{e}_3 \mathbf{v}_t}{r_{st}} = \frac{(\mathbf{v}_t \cdot r_{st})\, r_{st}}{r_{st}^3}$$

Thus, finally

$$\mathbf{G}_{ts\ell} = \int \varepsilon_0 \left(\mathbf{E}_s \times \mathbf{B}_t\right) d\tau = \frac{q_s q_t}{4\pi\varepsilon_0 c^2}\frac{1}{2}\left[\frac{\mathbf{v}_t}{r_{st}} + \frac{(\mathbf{v}_t \cdot \mathbf{r}_{st})\,\mathbf{r}_{st}}{r_{st}^3}\right] \tag{9.59}$$

and therefore

$$\mathbf{G}_{st\ell} = \int \varepsilon_0 \left(\mathbf{E}_t \times \mathbf{B}_s\right) d\tau = \frac{q_s q_t}{4\pi\varepsilon_0 c^2}\frac{1}{2}\left[\frac{\mathbf{v}_s}{r_{st}} + \frac{(\mathbf{v}_s \cdot \mathbf{r}_{st})\,\mathbf{r}_{st}}{r_{st}^3}\right] \tag{9.60}$$

When q_s, q_t are multivectors, belonging to a different algebra, the above would be written:

$$\mathbf{G}_{ts\ell} = \frac{1}{4\pi\varepsilon_0 c^2}\frac{1}{2}\left[\frac{\mathbf{v}_t}{r_{st}} + \frac{(\mathbf{v}_t \cdot \mathbf{r}_{st})\,\mathbf{r}_{st}}{r_{st}^3}\right]\frac{(q_t q_s + q_s q_t)}{2}$$

$$\mathbf{G}_{st\ell} = \frac{1}{4\pi\varepsilon_0 c^2}\frac{1}{2}\left[\frac{\mathbf{v}_s}{r_{st}} + \frac{(\mathbf{v}_s \cdot \mathbf{r}_{st})\,\mathbf{r}_{st}}{r_{st}^3}\right]\frac{(q_s q_t + q_t q_s)}{2}$$

9.3 Part III. Gravity

The preceding equations apply to electromagnetism. The electric field points away from its source, a positive charge, q_s. q_t is regarded as a negative charge so the force between q_s and q_t is attractive. One must keep this fact in mind.

If q_s and q_t are regarded as charges associated with gravitation masses m_s and m_t then the "charge" q_s and q_t are written $m_s q_s$ and $m_t q_t$. Also the force between them is attractive (negative). The direct **E** force on a charge $m_t q_t$ by a moving charge $m_s q_s$ is given by

$$\mathbf{F}_{ts}^{\mathrm{dir}} = \frac{1}{4\pi\varepsilon_0 c^2}\left[\frac{\mathbf{r}_{st}}{r^3} + \frac{1}{2c^2}\left(\frac{v_s^2 r_{st}}{r^3} + \frac{3\left(\mathbf{v}_s \cdot \mathbf{r}_{st}\right)^2 \mathbf{r}_{st}}{r^2} + \frac{1}{2c^2}\left(\frac{\mathbf{a}_s}{r} - \frac{\left(\mathbf{a}_s \cdot \mathbf{r}_{st}\right)\mathbf{r}_{st}}{r^3}\right)\right)\right]\left(m_s q_s m_t q_t + m_t q_t m_s q_s\right)/2 \tag{9.61}$$

It is understood that q_s is multiplied by m_s and q_t by m_t. We omit the display here on of m_s and m_t for brevity.

An independent calculation of the total angular momentum, $\mathbf{G}_a$, about an arbitrary point C, the position vector of which measured from $q_s\mathbf{v}_s$ is $\mathbf{R}_s$ and measured from $q_t\mathbf{v}_t$ is $\mathbf{R}_t$, gives:

$$\mathbf{G}_a = \frac{1}{4\pi\varepsilon_0 c^2}\left[\mathbf{R}_t \times \overbrace{\frac{1}{2}\left(\frac{\mathbf{v}_s}{r} + \frac{\left(\mathbf{v}_s \cdot \mathbf{r}_{st}\right) r_{st}}{r^3}\right)}^{\text{indirect force on } q_s} + \mathbf{R}_s \times \overbrace{\frac{1}{2}\left(\frac{\mathbf{v}_t}{r} + \frac{\left(\mathbf{v}_t \cdot \mathbf{r}_{st}\right)\mathbf{r}_{st}}{r^3}\right)}^{\text{indirect force on } q_t}\right]\left(q_s q_t + q_t q_s\right)/2 \tag{9.62}$$

The first term in Eq. (9.62) gives the total angular momentum in the field associated with lever arm $\mathbf{R}_t$. The second term gives the portion of the total angular momentum about C associated with the lever arm $\mathbf{R}_s$.

The total torque about P delivered to the field is

$$\begin{aligned}\frac{d\mathbf{G}_a}{dt} &= \mathbf{R}_t \times \text{force on } q_s\mathbf{v}_s + \mathbf{R}_s \times \text{force on } q_t\mathbf{v}_t \\ &= \mathbf{R}_t \times \left[\frac{d}{dt}\int \varepsilon_0\left(\mathbf{E}_t \times \mathbf{B}_s\right) d\tau \left(q_s q_t + q_t q_s\right)/2 \right. \qquad (9.63) \\ &\quad \left. + \mathbf{R}_s \times \frac{d}{dt}\int \varepsilon_0\left(\mathbf{E}_s \times \mathbf{B}_t\right) d\tau \left(q_s q_t + q_t q_s\right)/2 \right] \\ &= \left[\mathbf{R}_t \times \frac{d}{dt}\overset{\text{force on } q_s\mathbf{v}_s}{\mathbf{G}_\ell^{\varepsilon_0(\mathbf{E}_t\times\mathbf{B}_s)}} + \mathbf{R}_s \times \frac{d}{dt}\overset{\text{force on } q_t\mathbf{v}_t}{\mathbf{G}_\ell^{\varepsilon_0(\mathbf{E}_s\times\mathbf{B}_t)}}\right]\left(q_s q_t + q_t q_s\right)/2 \qquad (9.64)\end{aligned}$$

9.4 Summary. Direct Torques

$$\begin{aligned}& T_{ts}^{\mathbf{R}_t\times\mathbf{F}_{ts}^B} \\ &= \mathbf{R}_t \times \mathbf{F}_{ts}^B = \frac{1}{4\pi\varepsilon_0}\mathbf{R}_t \times \left[-\frac{2}{2c^2}\left(\frac{\mathbf{v}_t \times \left(\mathbf{v}_s \times \mathbf{r}_{st}\right)}{r^3}\right)\right]\left(q_s q_t + q_t q_s\right)/2 \\ &= \frac{1}{4\pi\varepsilon_0}\mathbf{R}_t \times \frac{1}{2c^2}\left[\frac{-2\mathbf{v}_t\left(\mathbf{v}_s \cdot \mathbf{r}_s\right) + 2\left(\mathbf{v}_t \cdot \mathbf{v}_s\right)\mathbf{r}_{st}}{r^3}\right]\left(q_s q_t + q_t q_s\right)/2\end{aligned}$$

– –

$$\begin{aligned}& T_{ts}^{\mathbf{R}_t\times\mathbf{F}_{ts}^E} \\ &= \mathbf{R}_t \times \mathbf{F}_{ts}^E \\ &= \frac{1}{4\pi\varepsilon_0}\left[-\frac{\mathbf{r}_{st}}{r^3} - \frac{1}{2c}\left(\frac{v_s^2\mathbf{r}_{st}}{r^3} + \frac{3\left(\mathbf{v}_s \cdot \mathbf{r}_{st}\right)^2\mathbf{r}_{st}}{r^5} + \frac{\mathbf{a}_s}{r} + \frac{\left(\mathbf{a}_s \cdot \mathbf{r}_{st}\right)\mathbf{r}_{st}}{r^3}\right)\right]\left(q_s q_t + q_t q_s\right)/2\end{aligned}$$

9.5 Indirect Torques

$$\begin{aligned}T_{ts}^{\varepsilon_0\frac{e}{dt}\int(\mathbf{E}_t\times\mathbf{B}_s)d\tau} &= \mathbf{R}_t \times \mathbf{F}_{ts}^{\varepsilon_0\frac{d}{dt}\int(\mathbf{E}_t\times\mathbf{B}_s)d\tau} = \mathbf{R}_t \times \frac{d}{dt}\mathbf{G}_{\ell ts} \\ &= -\frac{1}{4\pi\varepsilon_0 c^2}\left[\mathbf{R}_t \times \frac{d}{dt}\int\left(\frac{\mathbf{v}_s}{r} + \frac{\mathbf{v}_s \cdot \mathbf{r}_{st}}{r^3}\right) d\tau\right]\left(q_s q_t + q_t q_s\right)/2 \qquad (9.65)\end{aligned}$$

See Section 10.8 for the calculation of the derivative in Eq. (9.65).

One might add that we would be surprised if the torque about P were the result of crossing $\mathbf{R}_t\times$ force on q_t which would be the same form as the torque from a direct force on q_t.

9.6 Details Describing Conversion to Dimensionless Variables

Start with $\mathbf{r}_s$and $\mathbf{r}_{st}$ and divide $\mathbf{r}_s + \mathbf{r}_{st}$ by $\mathbf{r}_{st}$.

$$\frac{r_s}{r_{st}} + 1 = \rho_s + 1$$

We are expressing distances in terms of $\mathbf{r}_{st}$

$$\begin{aligned}
\int_{r_s - r_{st}}^{r_s + r_{st}} \frac{dr_t}{r_t^2} &= \int_{r_{st}\left(\frac{r_s}{r_{st}}-1\right)}^{r_{st}\left(\frac{r_s}{r_{st}}+1\right)} \frac{dr_t}{r_t^2} = \int_{r_{st}(\rho_s-1)}^{r_{st}(\rho_s+1)} \frac{dr_t}{r_t^2} \\
&= -\left[\frac{1}{r_t}\right] = -\left[\frac{1}{r_{st}\rho_t}\right]_{r_{st}\left(\frac{r_s}{r_{st}}-1\right)}^{r_{st}\left(\frac{r_s}{r_{st}}+1\right)} = -\left[\frac{1}{r_{st}\rho_t}\right]_{r_{st}(\rho-1)}^{r_{st}(\rho+1)} \\
&= -\left[\frac{1}{r_{st}\left(\rho_s+1\right)} - \frac{1}{r_{st}\left(\rho_s-1\right)}\right] = -\frac{1}{r_{st}}\left[\frac{1}{\rho_s+1} - \frac{1}{\rho_s-1}\right] \\
&= -\frac{1}{r_{st}}\left[\frac{(\rho_s-1)-(\rho_s+1)}{\rho_s^2-1}\right] = \frac{1}{r_{st}}\left[\frac{2}{\rho_s^2-1}\right]
\end{aligned}$$