6

REVERSION

It is shown that three of the multivectors of space-time algebra must have imaginary coefficients in order to represent conserved quantities, that is, in order to satisfy a continuity equation. The complex coefficients introduced herein have been simplified *ab initio* to pure imaginary coefficients by M. F. Ross (Ross 1980). However, it is interesting to begin with complex coefficients and show explicitly how imaginary coefficients suffice. We follow that approach here.

It will be shown that a general multivector M must have the form:

$$M = i\text{scalar} + \text{vector} + \text{biv} + i\text{triv} + i\text{quadvector}.$$

Consider the differential equation

$$\Box\mathcal{L} = m\mathcal{L}$$

where $\mathcal{L}$ is one, some, or all of the multivectors of space-time algebra.

$$\begin{aligned}\mathcal{L} &= \text{scalar} + \text{vector} + \text{bivector} + \text{trivector} + \text{quadvector}\\ &= S + \mathbf{v} + \mathbf{B} + \mathbf{T} + \mathbf{Q} \qquad\qquad \mathbf{Q} = \mathbf{e}_5 S'\end{aligned}$$

The gradient operator is

$$\Box = -\mathbf{e}_0\frac{\partial}{c\partial t} + \mathbf{e}_1\frac{\partial}{\partial x} + \mathbf{e}_2\frac{\partial}{\partial y} + \mathbf{e}_3\frac{\partial}{\partial z}$$

$\Box^2$ is called the d'Alembertion. It is covariant (retains its form) under a Lorentz transformation, so that $\Box^2 = \Box'^2$.

$$\Box^2 = -\frac{\partial^2}{c^2\partial t^2} + \frac{\partial}{\partial x^2} + \frac{\partial}{\partial y^2} + \frac{\partial}{\partial z^2}$$

In order to derive a conservation equation for the quantities in $\mathcal{L}$, Greider (1984, 473) defines the operation as a "complex reversion": "take the complex conjugate of the complex coefficients that multiply each of the 16 elements in the algebra, reverse the order of multiplication of the vectors in each multivector and in the $(-+++)$ metric reverse the signs of all vectors. (There is no sign change for $(+---)$.. . . Complex reversion is denoted by a tilde over the multivector" Greider (1984, 473).

$$\begin{aligned} \widetilde{\Box} &= -\widetilde{\mathbf{e}}_0\frac{\partial}{c\partial t} + \widetilde{\mathbf{e}}_1\frac{\partial}{\partial x} + \widetilde{\mathbf{e}}_2\frac{\partial}{\partial y} + \widetilde{\mathbf{e}}_3\frac{\partial}{\partial z} \\ &= \mathbf{e}_0\frac{\partial}{c\partial t} - \mathbf{e}_1\frac{\partial}{\partial x} \\ \widetilde{\Box} &= -\left[-\mathbf{e}_0\frac{\partial}{c\partial t} + \mathbf{e}_1\frac{\partial}{\partial x}\right] \\ &= -\Box \end{aligned}$$

When forming the complex reversion of the product of two groups, reverse the order of the groups, the form complex reversion of each group.

Start with

$$\Box\mathcal{L} = m\mathcal{L} \tag{6.1}$$

For clarity, when needed, or for emphasis, an arrow will point to the term to which $\Box$ is applied.

Perform reversion on Eq. (6.1)

$$\begin{aligned} \widetilde{\overset{\curvearrowright}{\Box}\mathcal{L}} &= \overset{\curvearrowleft}{\widetilde{\mathcal{L}}}\,\widetilde{\Box} = -\overset{\curvearrowleft}{\widetilde{\mathcal{L}}}\Box = m\widetilde{\mathcal{L}} \\ -\overset{\curvearrowleft}{\widetilde{\mathcal{L}}}\Box &= m\widetilde{\mathcal{L}} \end{aligned} \tag{6.2}$$

Multiply Eq. (6.1) on the left by $\widetilde{\mathcal{L}}$ to obtain

$$\widetilde{\mathcal{L}}\overset{\curvearrowright}{\Box}\mathcal{L} = m\widetilde{\mathcal{L}}\mathcal{L} \tag{6.3}$$

multiply Eq. (6.2) on the right by $\mathcal{L}$

$$-\overset{\curvearrowleft}{\widetilde{\mathcal{L}}}\Box\mathcal{L} = m\widetilde{\mathcal{L}}\mathcal{L} \tag{6.4}$$

Subtract Eq. (6.4) from Eq. (6.3)

$$\widetilde{\mathcal{L}}\overset{\curvearrowright}{\Box}\mathcal{L} + \overset{\curvearrowleft}{\widetilde{\mathcal{L}}}\Box\mathcal{L} = m\widetilde{\mathcal{L}}\mathcal{L} - m\widetilde{\mathcal{L}}\mathcal{L} = 0 \tag{6.5}$$

$$\Box = -\mathbf{e}_0\frac{\partial}{c\partial t} + \mathbf{e}_1\frac{\partial}{\partial x} + \mathbf{e}_2\frac{\partial}{\partial y} + \mathbf{e}_3\frac{\partial}{\partial z}$$

$$\Box = \mathbf{e}_0^{-1}\frac{\partial}{c\partial t} + \mathbf{e}_1\frac{\partial}{\partial x} + \mathbf{e}_2\frac{\partial}{\partial y} + \mathbf{e}_3\frac{\partial}{\partial z} \qquad \text{since } \mathbf{e}_0^{-1} = -\mathbf{e}_0,\ \mathbf{e}_1^{-1} = \mathbf{e}_1.\ \text{etc.}$$

Therefore Eq. (6.5) can be written explicitly

$$\widetilde{\mathcal{L}}\mathbf{e}_0^{-1}\frac{\overset{\curvearrowright}{\partial}}{c\partial t}\mathcal{L} + \overset{\curvearrowleft}{\mathcal{L}}\mathbf{e}_0^{-1}\frac{\partial}{c\partial t}\mathcal{L} + \widetilde{\mathcal{L}}\mathbf{e}_1^{-1}\frac{\overset{\curvearrowright}{\partial}}{\partial x}\mathcal{L} + \overset{\curvearrowleft}{\widetilde{\mathcal{L}}}\mathbf{e}_1^{-1}\frac{\partial}{\partial x}\mathcal{L}$$
$$+\widetilde{\mathcal{L}}\mathbf{e}_2^{-1}\frac{\overset{\curvearrowright}{\partial}}{\partial y}\mathcal{L} + \overset{\curvearrowleft}{\widetilde{\mathcal{L}}}\mathbf{e}_2^{-1}\frac{\partial}{\partial y}\mathcal{L} + \widetilde{\mathcal{L}}\mathbf{e}_3^{-1}\frac{\overset{\curvearrowright}{\partial}}{\partial z}\mathcal{L} + \overset{\curvearrowleft}{\widetilde{\mathcal{L}}}\mathbf{e}_3^{-1}\frac{\partial}{\partial z}\mathcal{L} = 0 \tag{6.6}$$

Integrate both sides over all space. Transform the space derivative terms to a surface integral by the divergence theorem and put the result equal to zero.

Eq. (6.6) may then be written

$$\frac{\partial}{c\partial t}\int\left(\widetilde{\mathcal{L}}\mathbf{e}_0^{-1}\mathcal{L}\right)dV \tag{6.7}$$

We now examine the nature of the terms forming

$$\widetilde{\mathcal{L}}\mathbf{e}_0^{-1}\mathcal{L}$$

The terms composing the integrand can vary from 1 to 16. We will consider the 16 *in toto* when defining the coefficients. We initially allowed them to be complex coefficients rather than merely imaginary coefficients and will follow that procedure here. However the results obtained from complex coefficients may be readily converted to imaginary coefficients and the final result remains unchanged, namely that for a space-time multivector to satisfy a continuity equation it must be of the form

$$M = i\text{scalar} + \text{vector} + \text{Biv} + i\text{Triv} + i\text{Quadvector}$$

6.1 Evaluation of $\widetilde{\mathcal{L}}e_0^{-1}\mathcal{L}$ Assuming Complex Coefficients for the Multivectors in $\mathcal{L}$

In the following, when evaluating $\widetilde{\mathcal{L}}\mathbf{e}_0^{-1}\mathcal{L}$, we consider $\mathcal{L}$ to have complex coefficients. It is concluded that three of the five multivectors comprising $\mathcal{L}$ must have imaginary coefficients in order that each be conserved, that is, that each satisfy a continuity equation. Thus $\mathcal{L}$ consists of any or all of the terms.

$$\mathcal{L} = \underset{\textbf{imaginary}}{i\text{scalar}} + \underset{\textbf{real}}{\text{vector}} + \underset{\textbf{real}}{\text{bivector}} + i\,\underset{\textbf{imaginary}}{\text{trivector}} + i\,\underset{\textbf{imaginary}}{\text{quadvector}} \tag{6.8}$$

It is also concluded that by allowing the coefficients to be simply imaginary *ab initio* rather than complex, the same three terms are intrinsically imaginary as shown in Eq. (6.8). We now carry out details assuming complex coefficients since we originally adopted this procedure.

6.2 Evaluation of $\widetilde{\mathcal{L}}e_0^{-1}\mathcal{L}$ Complex Coefficients

Begin with a complete set of all 16 multivectors of space-time algebra.

$$\begin{aligned}\mathcal{L} \;=\; &(S + v_0\mathbf{e}_0 + v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3 \\ &+B_{01}\mathbf{e}_0\mathbf{e}_1 + B_{02}\mathbf{e}_0\mathbf{e}_2 + B_{03}\mathbf{e}_0\mathbf{e}_3 + B_{12}\mathbf{e}_1\mathbf{e}_2 + B_{31}\mathbf{e}_3\mathbf{e}_1 + B_{23}\mathbf{e}_2\mathbf{e}_3 \\ &+T_{012}\mathbf{e}_0\mathbf{e}_1\mathbf{e}_2 + T_{031}\mathbf{e}_0\mathbf{e}_3\mathbf{e}_1 + T_{023}\mathbf{e}_0\mathbf{e}_2\mathbf{e}_3 + T_{123}\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 + S'\mathbf{e}_0\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3)\end{aligned} \tag{6.9}$$

all coefficients of $\mathcal{L}$ are complex. Recipe for reversion of $\mathcal{L}$ denoted by $\widetilde{\mathcal{L}}$ is:

Reverse sign and order of each multivector in $\mathcal{L}$ and change coefficients to their complex conjugates, that is, each $i \to i^* = -i$. $i = \sqrt{-1}$.

Form the reversion of Eq. (6.9) using Eq. (6.8):

$$\begin{aligned}\widetilde{\mathbf{e}_0\mathbf{e}_1} &= \mathbf{e}_1\mathbf{e}_0 = -\mathbf{e}_0\mathbf{e}_1, \quad \text{etc.} \qquad \widetilde{\mathbf{e}_1\mathbf{e}_2} = \mathbf{e}_2\mathbf{e}_1 = -\mathbf{e}_1\mathbf{e}_2, \quad \text{etc.} \\ \widetilde{\mathbf{e}_0\mathbf{e}_1\mathbf{e}_2} &= -\mathbf{e}_2\mathbf{e}_1\mathbf{e}_0 = -\mathbf{e}_0\mathbf{e}_2\mathbf{e}_1 = \mathbf{e}_0\mathbf{e}_1\mathbf{e}_2, \quad \text{etc.} \\ \widetilde{\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3} &= -\mathbf{e}_3\mathbf{e}_2\mathbf{e}_1 = -\mathbf{e}_1\mathbf{e}_3\mathbf{e}_2 = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 \\ \widetilde{\mathbf{e}_5} &= \widetilde{\mathbf{e}_0\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3} = \mathbf{e}_3\mathbf{e}_2\mathbf{e}_1\mathbf{e}_0 = -\mathbf{e}_0\mathbf{e}_3\mathbf{e}_2\mathbf{e}_1 = -\mathbf{e}_0\mathbf{e}_2\mathbf{e}_1\mathbf{e}_3 = \mathbf{e}_0\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\end{aligned}$$

Therefore

$$\begin{aligned}\widetilde{\mathcal{L}} \;=\; &(S^* - v_0^*\mathbf{e}_0 - v_1^*\mathbf{e}_1 - v_2^*\mathbf{e}_2 - v_3^*\mathbf{e}_3 \\ &-B_{01}^*\mathbf{e}_0\mathbf{e}_1 - B_{02}^*\mathbf{e}_0\mathbf{e}_2 - B_{03}^*\mathbf{e}_0\mathbf{e}_3 - B_{12}^*\mathbf{e}_1\mathbf{e}_2 - B_{31}^*\mathbf{e}_3\mathbf{e}_1 - B_{23}^*\mathbf{e}_2\mathbf{e}_3 \\ &+T_{012}^*\mathbf{e}_0\mathbf{e}_1\mathbf{e}_2 + T_{031}^*\mathbf{e}_0\mathbf{e}_3\mathbf{e}_1 + T_{023}^*\mathbf{e}_0\mathbf{e}_2\mathbf{e}_3 + T_{123}^*\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 + S'^*\mathbf{e}_5)\end{aligned} \tag{6.10}$$

Thus, in general

$$\begin{aligned}
\widetilde{\mathcal{L}}\mathbf{e}_\mu^{-1}\mathcal{L} \;=\; & (S^* - v_0^*\mathbf{e}_0 - v_1^*\mathbf{e}_1 - v_2^*\mathbf{e}_2 - v_3^*\mathbf{e}_3 \\
& -B_{01}^*\mathbf{e}_0\mathbf{e}_1 - B_{02}^*\mathbf{e}_0\mathbf{e}_2 - B_{03}^*\mathbf{e}_0\mathbf{e}_3 - B_{12}^*\mathbf{e}_1\mathbf{e}_2 - B_{31}^*\mathbf{e}_3\mathbf{e}_1 - B_{23}^*\mathbf{e}_2\mathbf{e}_3 \\
& +T_{012}^*\mathbf{e}_0\mathbf{e}_1\mathbf{e}_2 + T_{031}^*\mathbf{e}_0\mathbf{e}_3\mathbf{e}_1 + T_{023}^*\mathbf{e}_0\mathbf{e}_2\mathbf{e}_3 + T_{123}^*\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 + S'^*\mathbf{e}_5) \\
& \mathbf{e}_\mu^{-1}\,(S + v_0\mathbf{e}_0 + v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3 \\
& +B_{01}\mathbf{e}_0\mathbf{e}_1 + B_{02}\mathbf{e}_0\mathbf{e}_2 + B_{03}\mathbf{e}_0\mathbf{e}_3 + B_{12}\mathbf{e}_1\mathbf{e}_2 + B_{31}\mathbf{e}_3\mathbf{e}_1 + B_{23}\mathbf{e}_2\mathbf{e}_3 \\
& +T_{012}\mathbf{e}_0\mathbf{e}_1\mathbf{e}_2 + T_{031}\mathbf{e}_0\mathbf{e}_3\mathbf{e}_1 + T_{023}\mathbf{e}_0\mathbf{e}_2\mathbf{e}_3 + T_{123}\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 + S'\mathbf{e}_0\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3)
\end{aligned} \tag{6.11}$$

We now evaluate $\widetilde{\mathcal{L}}\mathbf{e}_0^{-1}\mathcal{L}$ which is then:

$$\begin{aligned}
\widetilde{\mathcal{L}}\mathbf{e}_0^{-1}\mathcal{L} \;=\; & -\widetilde{\mathcal{L}}\mathbf{e}_0\mathcal{L} = -(S^* - v_0^*\mathbf{e}_0 - v_1^*\mathbf{e}_1 - v_2^*\mathbf{e}_2 - v_3^*\mathbf{e}_3 \\
& -B_{01}^*\mathbf{e}_0\mathbf{e}_1 - B_{02}^*\mathbf{e}_0\mathbf{e}_2 - B_{03}^*\mathbf{e}_0\mathbf{e}_3 - B_{12}^*\mathbf{e}_1\mathbf{e}_2 - B_{31}^*\mathbf{e}_3\mathbf{e}_1 - B_{23}^*\mathbf{e}_2\mathbf{e}_3 \\
& +T_{012}^*\mathbf{e}_0\mathbf{e}_1\mathbf{e}_2 + T_{031}^*\mathbf{e}_0\mathbf{e}_3\mathbf{e}_1 + T_{023}^*\mathbf{e}_0\mathbf{e}_2\mathbf{e}_3 + T_{123}^*\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 + S'^*\mathbf{e}_5) \\
& \mathbf{e}_0\,(S + v_0\mathbf{e}_0 + v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3 \\
& +B_{01}\mathbf{e}_0\mathbf{e}_1 + B_{02}\mathbf{e}_0\mathbf{e}_2 + B_{03}\mathbf{e}_0\mathbf{e}_3 + B_{12}\mathbf{e}_1\mathbf{e}_2 + B_{31}\mathbf{e}_3\mathbf{e}_1 + B_{23}\mathbf{e}_2\mathbf{e}_3 \\
& +T_{012}\mathbf{e}_0\mathbf{e}_1\mathbf{e}_2 + T_{031}\mathbf{e}_0\mathbf{e}_3\mathbf{e}_1 + T_{023}\mathbf{e}_0\mathbf{e}_2\mathbf{e}_3 + T_{123}\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 + S'\mathbf{e}_0\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3)
\end{aligned} \tag{6.12}$$

We have after collecting the scalar terms

$$\begin{aligned}
& -v_0^*S - T_{123}^*S' - v_1^*B_{01} - v_2^*B_{02} - v_3^*B_{03} \\
& +v_0S^* + T_{123}S'^* + v_1B_{01}^* + v_2B_{02}^* + v_3B_{03}^* \\
& +B_{12}^*T_{012} + B_{31}^*T_{013} + B_{23}^*T_{023} \\
& -B_{12}T_{012}^* - B_{31}T_{013}^* - B_{23}T_{023}^*
\end{aligned} \tag{6.13}$$

These terms, in pairs, are of the form (using the first term in each line as an example). The scalar terms only appear multiplied by a vector coefficient. Thus

$$\begin{aligned}
& \quad S^* \qquad\quad v_0 \qquad\qquad S \qquad\quad v_0^* \\
& (a+ib)^*(c+id) - (a+ib)(c+id)^* \\
= \; & (a-ib)(c+id) - (a+ib)(c-id) \\
= \; & ac + bd + i(ad-bc) - [(ac+bd) - i(ad-bc)] \\
= \; & -i(ad-bc)2
\end{aligned} \tag{6.14}$$

Therefore, the scalar terms in Eq. (6.14) are pure imaginary. This means that the product of a scalar S and vector $\mathbf{v}$ is necessary to form a conservation equation. If

S is simply imaginary rather than complex, then its coefficient has the form ib and the coefficient of v_0 is real. In this case,

$$(ib)^* c - ibc - ibc - ibc = -ibc = \text{imaginary}$$

Thus an imaginary scalar coefficient combined with a real vector coefficient gives an imaginary result.

6.3 Vector Terms

$$\begin{aligned}
&-\mathbf{e}_0(S^*S + S'^*S' + v_0^*v_0 + v_1^*v_1 + v_2^*v_2 + v_3^*v_3\\
&+B_{01}^*B_{01} + B_{02}^*B_{02} + B_{03}^*B_{03} + B_{23}^*B_{23} + B_{31}^*B_{31} + B_{12}^*B_{12}\\
&+T_{012}^*T_{012} + T_{031}^*T_{031} + T_{023}^*T_{023} + T_{123}^*T_{123}
\end{aligned}$$

$$\begin{aligned}
&-\mathbf{e}_1(-S^*B_{01} - B_{23}^*S' + v_0^*v_1 - v_2^*T_{012} + v_3^*T_{031} + T_{023}^*T_{123} + B_{02}^*B_{12} - B_{03}^*B_{31}^*\\
&\qquad - SB_{01}^* - B_{23}S'^* - v_0v_1^* - v_2T_{012}^* + v_3T_{031}^* + T_{023}T_{123}^* + B_{02}B_{12}^* - B_{03}B_{31}^*)\\
&-\mathbf{e}_2(-S^*B_{02} - B_{31}^*S' + v_s^*v_2 + v_1^*T_{012} - v_3^*T_{023} + T_{013}^*T_{123} - B_{01}^*B_{12} + B_{03}^*B_{23}\\
&\quad -SB_{02}^* - B_{31}S'^* + v_0v_2^* + v_1T_{012}^* - v_3T_{023} + T_{013}T_{123}^* - B_{01}B_{12}^* + B_{03}B_{23}^*)\\
&-\mathbf{e}_3(-S^*B_{03} - B_{12}^*S' + v_0^*v_3 - v_1^*T_{031} + v_2^*T_{023} + T_{012}^*T_{123} + B_{01}^*B_{31} - B_{02}^*B_{23}\\
&\qquad - SB_{03}^* - B_{12}S'^* + v_0v_3^* - v_1T_{031}^* + v_2T_{023} + T_{012}T_{123}^* + B_{01}B_{31}^* - B_{02}B_{23}^*)
\end{aligned}$$

These coefficients in pairs are all of the form, taking the coefficient of $\mathbf{e}_3$ as an example

$$\begin{aligned}
S^*B_{03} + SB_{03}^* &= (a - ib)(c + id) + (a + ib)(c - id)\\
&= ac + iad - ibc + bd + ac - iad + ibc + bd = (2ac + bd)
\end{aligned}$$

Therefore, the sum of the coefficients of the vector terms is real. Consider the coefficient of $\mathbf{e}_0$.

$$-\mathbf{e}_0\,(S^*S) = -\mathbf{e}_0\,(a - ib)\,(a + ib) = -\mathbf{e}_0\left(a^2 + b^2\right) = \text{real}$$

The same conclusion holds for other pairs in line one and also for lines two and three. For example, in line two

$$-\mathbf{e}_0\,({B_{01}}^*B_{01}) = -\mathbf{e}_0\,(a - ib)\,(a + ib) = -\mathbf{e}_0\left(a^2 + b^2\right) = \text{real}$$

Consider the coefficients of $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ by vertical pairs. For example

$$\begin{aligned} -\mathbf{e}_3\left(-S^*B_{03}-SB_{03}^*\right) &= \mathbf{e}_3\left(SB_{03}^*+S^*B_{03}\right) \\ &= \mathbf{e}_3\left[(a+ib)(a-ib)+(a-ib)(a+ib)\right] \\ &= \mathbf{e}_3\left[\left(a^2+b^2\right)+a^2+b^2\right] \\ &= \mathbf{e}_3 2\left(a^2+b^2\right) = \text{ real} \end{aligned}$$

The conclusion is that all vector terms are real.

Hereafter, we will write the bivector coefficients in the familiar electromagnetic notation

$$\begin{aligned} & B_{01}\mathbf{e}_0\mathbf{e}_1+B_{02}\mathbf{e}_0\mathbf{e}_2+B_{03}\mathbf{e}_0\mathbf{e}_3+B_{23}\mathbf{e}_2\mathbf{e}_3+B_{31}\mathbf{e}_3\mathbf{e}_1+B_{12}\mathbf{e}_1\mathbf{e}_2 \\ = \; & E_x\mathbf{e}_0\mathbf{e}_1+E_y\mathbf{e}_0\mathbf{e}_2+E_z\mathbf{e}_0\mathbf{e}_3+B_x\mathbf{e}_2\mathbf{e}_3+B_y\mathbf{e}_3\mathbf{e}_1+B_z\mathbf{e}_1\mathbf{e}_2 \end{aligned}$$

even though the bivector coefficients are not necessarily **E** and **B** fields.

Also, the trivector will be written

$$\begin{aligned} & T_{123}\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3+T_{023}\mathbf{e}_0\mathbf{e}_2\mathbf{e}_3+T_{031}\mathbf{e}_0\mathbf{e}_3\mathbf{e}_1+T_{012}\mathbf{e}_0\mathbf{e}_1\mathbf{e}_2 \\ = \; & \mathbf{e}_5(u_0\mathbf{e}_0+\quad u_1\mathbf{e}_1+\quad u_2\mathbf{e}_2+\quad u_3\mathbf{e}_3) \end{aligned}$$

So $$u_0=T_{123},\qquad u_1=T_{023},\qquad u_2=T_{031},\qquad u_3=T_{012}$$

The previous set of scalar terms are then written

$$\begin{aligned} & -v_0^*S-u_0^*S'-v_1^*E_x-v_2^*E_y-v_3^*E_z\;+B_z^*u_3+B_y^*u_2+B_x^*u_1 \\ & +v_0S^*+u_0S'^*+v_1E_x^*+v_2E_y^*+v_3E_z^*-B_zu_3^*-B_yu_2^*-B_xu_1^* \end{aligned}$$

As shown earlier, the sum of any pair in vertical alignment is of the form:

$$\begin{aligned} v_0S^*-v_0^*S &= (a+ib)(c-id)-(a-ib)(c+id) \\ &= ac-iad+ibc+bd-ac-iad+ibc-bd \\ &= 2i(bc-ad) \end{aligned}$$

So the above group is pure imaginary. Also consider $v_0S^*-v_0^*S$ in the following way. v_0 is a scalar vector coefficient represented by a above. If S^* were real, then one would have $ac-ac=0$. If S^* is pure imaginary, than $S=id$. Then

$$v_0S^*-v_0^*S=a\,(id)^*-a\,(id)=iad$$

Thus all of the complex coefficients can be changed to real or imaginary coefficients from which one can deduce whether the multivector coefficients is real or imaginary. The conclusion that a complex coefficient may be replaced by an imaginary coefficient.

In the following, as before, we allow all coefficients to be complex. In each case, the complex coefficient may be replaced by an imaginary $i = \sqrt{-1}$ coefficient. The conclusion, however, remains the same whether the products of the various multivectors are real or imaginary.

6.4 Vector Terms. Real.

Coefficient of $\mathbf{e}_0$ positive definite. Likewise for the other coefficients.

$$\begin{aligned}
&-\mathbf{e}_0(S^*S + S'^*S' + \mathbf{v}^*\cdot\mathbf{v} + \mathbf{E}^*\cdot\mathbf{E} + \mathbf{B}^*\cdot\mathbf{B} + \mathbf{u}^*\cdot\mathbf{u}) \\
&-\mathbf{e}_1\{[S^*E_x + (S^*E_x)^* - [S'^*B_x + (S^*B_x)^*] - [\mathbf{v}^*\times\mathbf{u})_x + (\mathbf{v}^*\times\mathbf{u})_x^*] \\
&\quad +[v_0^*v_1 + (v_0^*v_1)^*] + [u_0^*u_1 + (u_0^*u_1)^*] + [(\mathbf{E}^*\times\mathbf{B})_x + (\mathbf{E}^*\times\mathbf{B})_x^*]\} \\
&-\mathbf{e}_2\{[S^*E_y + (S^*E_y)^*] - [S'^*B_y + (S'^*B_y)^*] - [(\mathbf{v}^*\times\mathbf{u})_y + \mathbf{v}^*\times\mathbf{u})_y^*] \\
&\quad + [v_0^*v_2 + (v_0^*v_2)^*] + [u_0^*u_2 + (u_0^*u_2)^*] + [(\mathbf{E}^*\times\mathbf{B})_y + (\mathbf{E}^*\times\mathbf{B})_y^*]\} \\
&-\mathbf{e}_3\{-[S^*E_z + (S^*E_z)^* -][S'^*B_z - (S'^*B_z)^*] - [(\mathbf{v}^*\times\mathbf{u})_z - (\mathbf{v}^*\times\mathbf{u})_z^*] \\
&\quad + [v_0^*v_3 + (v_0^*v_3)^*] + [u_0^*u_3 + (u_0^*u_3)^*] + [(\mathbf{E}^*\times\mathbf{B})_z + (\mathbf{E}^*\times\mathbf{B})_z^*]\}
\end{aligned}$$

6.5 Bivector Terms

All bivector terms are real because they have the form displayed below,

$$\begin{aligned}
(a+ib)(c-id) + (a-ib)(c+id) &= ac + bd + i(bc - ad) + (ac + bd) - i(bc - ad) \\
&= 2(ac + bd)
\end{aligned}$$

For example, for the first pair, $v_1 S^* + v_1^* S = 2\,(ac + bd) =$ real.

$$\begin{aligned}
&\mathbf{e}_0\mathbf{e}_1(-v_1^* S - v_0^* E_x + v_2^* B_z - v_3^* B_y + E_y^* u_3 - E_z^* u_2 - B_x^* u_0 \\
&\qquad - v_1 S^* - v_0 E_x^* + v_2 B_z^* - v_3 B_y^* + E_y u_3^* - E_z u_2^* - B_x u_0^*) \\
&\mathbf{e}_0\mathbf{e}_2(-v_2^* S - v_0^* E_y - v_1^* B_z + v_3^* B_x - E_x^* u_3 + E_z^* u_1 - B_y^* u_0 - u_2^* S' \\
&\qquad - v_2 S^* - v_0 E_y^* - v_1 B_z^* + v_3 B_x^* - E_x u_3^* + E_z u_1^* - B_y u_0^* - u_2 S'^*) \\
&\mathbf{e}_0\mathbf{e}_3(-v_3^* S - v_0 E_z + v_1^* B_y - v_2^* B_x + E_x^* u_2 - E_y^* u_1 - B_z^* u_0 - u_3^* S' \\
&-v_3 S^* - v_0 E_z^* + v_1 B_y^* - v_2 B_x^* + E_x u_2^* - E_y u_1^* - B_z u_0^* - u_3 S'^*) \\
&\mathbf{e}_1\mathbf{e}_2(-v_3^* S' - v_0^* B_z - v_1^* E_y + v_2 E_x - B_y u_1 + B_x^* u_2 + E_z u_0 + u_3^* S \\
&-v_3 S' - v_0 B_z^* - v_1 E_y^* + v_2 E_x^* - B_y u_1^* + B_x u_2^* + E_z u_0^* + u_3 S^*)
\end{aligned}$$

$$\begin{aligned}
&\mathbf{e}_3\mathbf{e}_1(-v_2^* S_x' - v_0^* B_y - v_1^* E_z - v_3^* E_x + B_z^* u_1 - B_x^* u_3 + E_y^* u_0 + u_2^* S \\
&- v_2 S'^* - v_0 B_y^* - v_1 E_z^* - v_3 E_x^* + B_z u_1^* - B_x u_3^* + E_y u_0^* + u_2 S^*) \\
&\mathbf{e}_2\mathbf{e}_3(-v_1^* S' - v_0^* B_x - v_2^* E_z + v_3^* E_y + B_y^* u_3 - B_z^* u_3 + E_x^* u_0 + u_1^* S \\
&-v_1 S'^* - v_0 B_x^* - v_2 E_z^* + v_3 E_y^* + B_y u_3^* - B_z u_3^* + E_x u_0^* + u_1 S^*)
\end{aligned}$$

Consider

$$\begin{aligned}
\mathbf{e}_0\mathbf{e}_1\,(-v_0^* B_x - v_0 B_x^*) &= -\mathbf{e}_0\mathbf{e}_1(ac + ac) \\
&= \mathbf{e}_0\mathbf{e}_1(2ac) = \text{ real}
\end{aligned}$$

6.6 Bivector Terms. Real.

$$+\mathbf{e}_0\mathbf{e}_1\{-[(v_1^*S)-(v_1^*S)^*]-[(u_1^*S')+(u_1^*S')^*]$$
$$-[(v_0^*E_x)+v_1^*E_x)^*]+[(\mathbf{E}^*\times\mathbf{u})_x+(\mathbf{E}^*\times\mathbf{u})_x^*]$$
$$-[u_0^*B_x+(u_0^*B_x)^*]-[(\mathbf{B}^*\times\mathbf{v})_x+(\mathbf{B}^*\times\mathbf{v})^*]\}$$
$$+\mathbf{e}_0\mathbf{e}_2\{-[(v_2^*S)+(v_2^*S)^*]-[(u_2^*S')+(u_2^*S')^*]$$
$$-[(v_0E_y)+(v_0^*E_y)^*]+[(\mathbf{E}^*\times\mathbf{u})_y+\mathbf{E}^*\times\mathbf{u})_y^*]$$
$$-[(u_0^*B_y+(u_0^*B_y)^*]-[(\mathbf{B}^*\times\mathbf{v})_y+(\mathbf{B}^*\times\mathbf{v})_y^*]\}$$
$$+\mathbf{e}_0\mathbf{e}_3\{-[(v_3^*S)+(v_3^*S)^*]-[(u_3^*S')+u_3^*S')^*]$$
$$-[(v_0E_z)+(v_0^*E_z)^*]+[(\mathbf{E}^*\times\mathbf{u})_z+(\mathbf{E}^*\times\mathbf{u})_z^*]$$
$$-[(u_0^*B_z)+(u_0^*B_z)^*]-[(\mathbf{B}^*\times\mathbf{v})_z+(\mathbf{B}^*\times v)_z^*]\}$$
$$+\mathbf{e}_1\mathbf{e}_2\{-[(v_3^*S')+(v_3^*S')^*]+[(u_3^*S)+(u_3^*S)^*]$$
$$+[(u_0^*E_z)+(u_0^*E_z)^*]+[(\mathbf{E}^*\times\mathbf{v})_z+(\mathbf{E}^*\times\mathbf{v})_z^*]$$
$$-[(v_0^*B_z)+(v_0B_z)^*]+[(\mathbf{B}^*\times\mathbf{u})_z+(\mathbf{B}^*\times\mathbf{u})_z^*]$$
$$+\mathbf{e}_3\mathbf{e}_1\{-[(v_2^*S')+(v_2^*S')^*]+[(u_2^*S)+(u_2^*S)^*]$$
$$+[(u_0^*E_y)+(u_0^*E_y)^*]+[(\mathbf{E}^*\times\mathbf{v})_y+(\mathbf{E}^*\times\mathbf{v})_y^*]$$
$$-[(v_0^*B_y)-(v_0^*B_y)^*]+[(\mathbf{B}^*\times\mathbf{u})_y+(\mathbf{B}^*\times\mathbf{u})_y^*]\}$$
$$+\mathbf{e}_2\mathbf{e}_3\{-[(v_1^*S')+(v_1^*S')^*+[(u_1^*S)+(u_1^*S)^*]$$
$$+[(u_0^*E_x)+(u_0^*E_x)^*]+[(\mathbf{E}^*\times\mathbf{v})_x+(\mathbf{E}^*\times\mathbf{v})_x^*]$$
$$-[v_0^*B_x+(v_0^*B_x)^*]+(\mathbf{B}^*\times\mathbf{u})_x+(\mathbf{B}^*\times\mathbf{u})_x^*]\}$$

6.7 Trivector Terms. Trivector terms are imaginary

$$\begin{aligned}
&\mathbf{e}_0\mathbf{e}_1\mathbf{e}_2(-S^*B_z - S'^*E_z - v_1^*v_2 - v_3^*u_0 - v_0^*u_3 + u_2^*u_1 - E_x^*E_y + B_y^*B_x \\
&\qquad + SB_z + S'E_z^* + v_1v_2^* + v_3u_0^* + v_0u_3^* - u_2u_1^* + E_xE_y^* - B_yB_x^*) \\
&\mathbf{e}_0\mathbf{e}_1\mathbf{e}_3(S^*B_y + S^*E_y - v_1^*v_3 + v_2^*u_0 - v_0^*u_2 + u_3^*u_1 - E_x^*E_z + B_z^*B_x \\
&\qquad - SB_y^* - S'E_y^* + v_1v_3^* - v_2u_0^* + v_0u_2^* - u_3u_1^* + E_xE_z^* - B_zB_x^*) \\
&\mathbf{e}_0\mathbf{e}_2\mathbf{e}_3(-S^*B_x + S_1'^*E_x - v_2^*v_3 - v_1^*u_0 - v_0^*u_1 + u_3^*u_2 - E_y^*E_z + B_z^*B_y \\
&\qquad + SB_x^* - S'E_x^* + v_2v_3^* + v_1u_0^* + v_0u_1^* - u_3u_2^* + E_yE_z^* - B_zB_y^*) \\
&\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3(S^*S' - v_2^*u_2 - v_1^*u_1 - v_0^*u_0 - v_3^*u_3 + E_x^*B_x + E_y^*B_y + E_z^*B_z \\
&\qquad - SS'^* + v_2u_2^* + v_1u_1^* + v_0u_0^* + v_3u_3^* - E_xB_x^* - E_yB_y^* + E_zB_z^*)
\end{aligned}$$

$$\begin{aligned}
& (a+ib) - (a+ib)^* \\
= \;& 2ib(B_z - iB_z')(S_1 + iS_2) - (B_z - iB_z')(S_1 + iS_2)^* \\
& \mathbf{e}_0\mathbf{e}_1\mathbf{e}_2\{[(B_z^*S) - (B_z^*S)^*] + [(E_z^*S') - (E_zS')^*] \\
& +[(v_2^*v_1) - (v_2^*v_1)^*] + [(u_0^*v_3) - (u_0^*v_3)^*] + [(u_3^*v_0) - (u_3^*v_0)^*] \\
& +[u_2^*u_1) - (u_2^*u_1)^*] \\
& +[(E_y^*E_x) - (E_y^*E_x)^*] + [(B_y^*B_x) - (B_y^*B_x)^*]\} \\
& \mathbf{e}_0\mathbf{e}_1\mathbf{e}_3\{[-(B_y^*S) - (B_y^*S)^*] - [(E_y^*S') - (E_y^*S')^*] \\
& +[(v_3^*v_1) - (v_3^*v_1)^*] - [(u_0^*v_2) - (u_0^*v_2)^*] + [(u_2^*v_0) - (u_2^*v_0)^*] \\
& +[(u_3^*u_1) - (u_3^*u_1)^*] \\
& +[(E_z^*E_x) - (E_z^*E_x)^*] + [(B_z^*B_x) - (B_z^*B_x)^*]\} \\
& +\mathbf{e}_0\mathbf{e}_2\mathbf{e}_3\{[(B_x^*S) - (B_x^*S)^*] - [(E_x^*S') - (E_x^*S'])^* \\
& -[(v_2^*v_3) - (v_2^*v_3)^*] + [(u_0^*v_1) - (u_0^*v_1)^*] + [(u_1^*v_0) - (u_1^*v_0)^*] \\
& +[(u_3^*u_2) - (u_3^*u_2)^*] \\
& +[(E_z^*E_y) - (E_z^*E_y)^*] + [(B_z^*B_y) - (B_z^*B_y)^*]\} \\
& +\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\{[(S^*S') - (S^*S')] + [(\mathbf{u}^*\cdot\mathbf{v}) - (\mathbf{u}^*\cdot\mathbf{v})^*] \\
& +[(\mathbf{E}^*\cdot\mathbf{B}) - (\mathbf{E}^*\cdot\mathbf{B})^*]\}
\end{aligned}$$

Example, coefficient of $\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$

$$\begin{aligned}
-\left(SB_y^* - S^*B_y\right) &= (a+ib)(c-id) - (a-ib)(c+id) \\
&= 2i(bc - ad)
\end{aligned}$$

All trivector terms are pure imaginary since they have the form

$$(a+ib)(c-id) - (a-ib)(c+id) = 2i(bc-ad)$$

6.8 Quadvector Terms. Imaginary.

$$\mathbf{e}_0\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3(S'^*v_0 - S^*u_0 - v_1^*B_x - v_2^*B_y - v_3^*B_z - E_x^*u_1 - E_y^*u_2 - E_z^*u_3 \\ - S'v_0^* + Su_0^* + v_1B_x^* + v_2B_y^* + v_3B_z^* + E_xu_1^* + E_yu_2^* + E_zu_3^*)$$

Observation: If both members of the above pairs are pure real or pure imaginary, the terms reduce to zero. Therefore, if one member of a pair is real, the other must be imaginary, not to vanish.

We write the terms over once more. If $S =$ imaginary and $v_0 =$ imaginary,

$$(iv_0)(-iS) - (iv_0)(-iS)^* = 0$$

$\therefore$ v_0 must be real for non-vanishing. Also, if $S' =$ imaginary, $u_0 =$ real.

$$\begin{aligned}(a+ib)+(a+ib)^* &= 2a \\ (a+ib)-(a+ib)^* &= 2ib\end{aligned}$$

6.9 Quadvector Term. Pure Imaginary

$$\mathbf{e}_0\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\{[(S'^*v_0) - (S'^*v_0)^*] - [(S^*u_0) - (S^*u_0)^*] \\ +[(\mathbf{u}^*\cdot\mathbf{E}) - (\mathbf{u}^*\cdot\mathbf{E})^*] - [(\mathbf{v}^*\cdot\mathbf{B}) - (\mathbf{v}^*\cdot\mathbf{B})^*] + [(\mathbf{E}\cdot\mathbf{u}^*) - (\mathbf{E}\cdot\mathbf{u}^*)^*]\}$$

6.10 Scalar Terms. Scalar terms are imaginary

$$[(v_0S^*) - (v_0S^*)^*] + [u_0S'^* - (u_0S'^*)^*] \\ +[(\mathbf{E}^*\cdot\mathbf{v}) - (\mathbf{E}^*\cdot\mathbf{v})^*] + [\mathbf{B}^*\cdot\mathbf{u}) - (\mathbf{B}^*\cdot\mathbf{u})^*]$$

If $\mathbf{v} =$ real and $\mathbf{E} =$ real, then these terms are zero.
If $S =$ imag. and $v_0 =$ imag.,

$$(iv_0)(-is) - (iv_0)(-is)^x = 0$$

Therefore v_0 must be real for non-vanishing. Also if $S' =$ imag., $v_0 =$ real

$$\begin{aligned}(a+ib)+(a+ib)^* &= 2a \\ (a+ib)-(a+ib)^* &= 2ib\end{aligned}$$

6.11 Summary

The 16 elements of space-time algebra are:

$$\begin{aligned}\mathcal{L} &= iS + v_0\mathbf{e}_0 + v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3 \\ &+\overset{E_x}{B_{01}\mathbf{e}_0\mathbf{e}_1} + \overset{E_y}{B_{02}\mathbf{e}_0\mathbf{e}_2} + \overset{E_z}{B_{03}\mathbf{e}_0\mathbf{e}_3} + \overset{B_z}{B_{12}\mathbf{e}_1\mathbf{e}_2} + \overset{B_y}{B_{31}\mathbf{e}_3\mathbf{e}_1} + \overset{B_x}{B_{23}\mathbf{e}_2\mathbf{e}_3} \\ &+i[T_{123}\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 + T_{023}\mathbf{e}_0\mathbf{e}_2\mathbf{e}_3 + T_{031}\mathbf{e}_0\mathbf{e}_3\mathbf{e}_1 + T_{012}\mathbf{e}_0\mathbf{e}_1\mathbf{e}_2] \\ &+ i\,\mathbf{e}_0\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 S'\end{aligned}$$

that is;

$$\mathcal{L} = \underset{\textbf{imaginary}}{i \text{ scalar}} \quad + \quad \underset{\textbf{real}}{\text{vector}} \; + \underset{\textbf{real}}{\text{bivector}} \; + \underset{\textbf{imaginary}}{i \text{ trivector}} \; + \underset{\textbf{imaginary}}{i \text{ quadvector}}$$

The trivector terms may be written

$$i\mathbf{e}_5(u_0\mathbf{e}_0 + u_1\mathbf{e}_1 + u_2\mathbf{e}_2 + u_3\mathbf{e}_3)$$

The coefficients of the scalar element, the trivector elements, and the quadvector elements are pure imaginary (must be multiplied by i). The coefficients of the vector elements and the bivector elements are real. These modifications are necessary in order that each element in the algebra satisfies a continuity equation.

$$\frac{\partial}{\partial x_\mu}\widetilde{\mathcal{L}}\mathbf{e}_u^{-1}\mathcal{L} = 0$$