

5

MAXWELL'S EQUATIONS

5.1 Derivation of Maxwell's Equations using Space-time Algebra Format

Consider the space-time vector

$$\mathbf{A} = \mathbf{e}_0 A_0 + \mathbf{e}_1 A_x + \mathbf{e}_2 A_y + \mathbf{e}_3 A_z = \mathbf{e}_0 A_0 + \mathbf{A}' = \mathbf{e}_0 \phi / c + \mathbf{a} \quad (5.1)$$

In the usual 3-space description of electromagnetism upper case \mathbf{A} is used to denote the spatial part of Eq. (5.1), that is, to denote what we call $\mathbf{e}_1 A_x + \mathbf{e}_2 A_y + \mathbf{e}_3 A_z$, a notation we would like to retain. However, we are now more interested in the total space time vector, so we label *it* \mathbf{A} . The last three terms of \mathbf{A} will be called \mathbf{a} or \mathbf{A}' when discussing the usual 3-space formulation of electromagnetism. To obtain what we will call the “equation of motion” of \mathbf{A} , form $\square \mathbf{A}$ then identify the result with a physical quantity that has the same geometry. Thus

$$\square \mathbf{A} = \left[-\mathbf{e}_0 \frac{\partial}{c \partial t} + \mathbf{e}_1 \frac{\partial}{\partial x} + \mathbf{e}_2 \frac{\partial}{\partial y} + \mathbf{e}_3 \frac{\partial}{\partial z} \right] (\mathbf{e}_0 A_0 + \mathbf{e}_1 A_x + \mathbf{e}_2 A_y + \mathbf{e}_3 A_z)$$

$$\begin{aligned}
\Box \mathbf{A} = & -\mathbf{e}_0 \mathbf{e}_0 \frac{\partial A_0}{c \partial t} - \mathbf{e}_0 \mathbf{e}_1 \frac{\partial A_x}{c \partial t} - \mathbf{e}_0 \mathbf{e}_2 \frac{\partial A_y}{c \partial t} - \mathbf{e}_0 \mathbf{e}_3 \frac{\partial A_z}{c \partial t} \\
& + \mathbf{e}_1 \mathbf{e}_0 \frac{\partial A_0}{\partial x} + \mathbf{e}_1 \mathbf{e}_1 \frac{\partial A_x}{\partial x} + \mathbf{e}_1 \mathbf{e}_2 \frac{\partial A_y}{\partial x} + \mathbf{e}_1 \mathbf{e}_3 \frac{\partial A_z}{\partial x} \\
& + \mathbf{e}_2 \mathbf{e}_0 \frac{\partial A_0}{\partial y} + \mathbf{e}_2 \mathbf{e}_1 \frac{\partial A_x}{\partial y} + \mathbf{e}_2 \mathbf{e}_2 \frac{\partial A_y}{\partial y} + \mathbf{e}_2 \mathbf{e}_3 \frac{\partial A_z}{\partial y} \\
& + \mathbf{e}_3 \mathbf{e}_0 \frac{\partial A_0}{\partial z} + \mathbf{e}_3 \mathbf{e}_1 \frac{\partial A_x}{\partial z} + \mathbf{e}_3 \mathbf{e}_2 \frac{\partial A_y}{\partial z} + \mathbf{e}_3 \mathbf{e}_3 \frac{\partial A_z}{\partial z}
\end{aligned} \tag{5.2}$$

$$\Box = \left(-\mathbf{e}_0 \frac{\partial}{c \partial t} + \mathbf{e}_1 \frac{\partial}{\partial x} + \mathbf{e}_2 \frac{\partial}{\partial y} + \mathbf{e}_3 \frac{\partial}{\partial z} \right) = -\mathbf{e}_0 \frac{\partial}{c \partial t} + \nabla = -\mathbf{e}_0 \frac{\partial}{c \partial t} + \text{grad}$$

$\Box^2 = \nabla^2 - \frac{\partial^2}{c^2 \partial t^2}$ is known as the d'Alembertian operator. It is invariant (covariant) under a Lorentz transformation so that $\Box^2 = \Box'^2$.

Rearranging terms:

$$\begin{aligned}
\Box \mathbf{A} = & \left[-\mathbf{e}_0 \mathbf{e}_0 \frac{\partial A_0}{c \partial t} + \mathbf{e}_0 \mathbf{e}_1 \frac{1}{2} \left(-\frac{\partial A_x}{c \partial t} - \frac{\partial A_0}{\partial x} \right) + \mathbf{e}_0 \mathbf{e}_2 \frac{1}{2} \left(-\frac{\partial A_y}{c \partial t} - \frac{\partial A_0}{\partial y} \right) \right. \\
& \quad \left. + \mathbf{e}_0 \mathbf{e}_3 \frac{1}{2} \left(-\frac{\partial A_z}{c \partial t} - \frac{\partial A_0}{\partial z} \right) \right. \\
& + \mathbf{e}_1 \mathbf{e}_0 \frac{1}{2} \left(\frac{\partial A_x}{c \partial t} + \frac{\partial A_0}{\partial x} \right) + \mathbf{e}_1 \mathbf{e}_1 \frac{\partial A_x}{\partial x} + \mathbf{e}_1 \mathbf{e}_2 \frac{1}{2} \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \\
& \quad \left. + \mathbf{e}_1 \mathbf{e}_3 \frac{1}{2} \left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z} \right) \right. \\
& + \mathbf{e}_2 \mathbf{e}_0 \frac{1}{2} \left(\frac{\partial A_y}{c \partial t} + \frac{\partial A_0}{\partial y} \right) + \mathbf{e}_2 \mathbf{e}_1 \frac{1}{2} \left(\frac{\partial A_x}{\partial y} - \frac{\partial A_y}{\partial x} \right) + \mathbf{e}_2 \mathbf{e}_2 \frac{\partial A_y}{\partial y} \\
& \quad \left. + \mathbf{e}_2 \mathbf{e}_3 \frac{1}{2} \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) \right. \\
& + \mathbf{e}_3 \mathbf{e}_0 \frac{1}{2} \left(\frac{\partial A_z}{c \partial t} + \frac{\partial A_0}{\partial z} \right) + \mathbf{e}_3 \mathbf{e}_1 \frac{1}{2} \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) + \mathbf{e}_3 \mathbf{e}_2 \frac{1}{2} \left(\frac{\partial A_y}{\partial z} - \frac{\partial A_z}{\partial y} \right) \\
& \quad \left. + \mathbf{e}_3 \mathbf{e}_3 \frac{\partial A_z}{\partial z} \right] q_s
\end{aligned} \tag{5.3}$$

which clearly shows that the coefficients of the bivectors which are composed of derivatives of the components of \mathbf{A} form an antisymmetric tensor of rank two. Rewriting Eq. (5.3)

$$\begin{aligned}
 \square \mathbf{A} = & \\
 & \left[\begin{aligned}
 & \begin{matrix} E_x/c \\ \mathbf{e}_0 \mathbf{e}_1 \left(-\frac{\partial A_x}{c \partial t} - \frac{\partial A_0}{\partial x} \right) \end{matrix} + \begin{matrix} E_y/c \\ \mathbf{e}_0 \mathbf{e}_2 \left(-\frac{\partial A_y}{c \partial t} - \frac{\partial A_0}{\partial y} \right) \end{matrix} + \begin{matrix} E_z/c \\ \mathbf{e}_0 \mathbf{e}_3 \left(-\frac{\partial A_z}{c \partial t} - \frac{\partial A_0}{\partial z} \right) \end{matrix} \\
 & \begin{matrix} B_z \\ \mathbf{e}_1 \mathbf{e}_2 \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \end{matrix} + \begin{matrix} B_y \\ \mathbf{e}_3 \mathbf{e}_1 \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) \end{matrix} \\
 & \begin{matrix} B_x \\ \mathbf{e}_2 \mathbf{e}_3 \left(\frac{\partial A_z}{\partial x} - \frac{\partial A_y}{\partial z} \right) \end{matrix} + \square \bullet \mathbf{A} \end{aligned} \right] q_s
 \end{aligned} \tag{5.4}$$

We use the large Clifford algebra dot, \bullet . It is the same as the Gibbs dot product, small dot \cdot for the scalar part of $\square \bullet \mathbf{A}$ when the direct product of two vectors is involved as is the case above. The scalar is:

$$\square \bullet \mathbf{A} = \frac{\partial A_0}{c \partial t} + \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \tag{5.5}$$

The bivector terms correspond to the combination $(\square \mathbf{A} - \mathbf{A} \square) / 2 \equiv \square \wedge \mathbf{A}$. The scalar term corresponds to $(\square \mathbf{A} + \mathbf{A} \square) / 2 \equiv \square \bullet \mathbf{A}$.

If \mathbf{A} is regarded as a potential, then it is reasonable to associate the derivatives of \mathbf{A} with forces. A_0 may be identified with the usual scalar electric potential, also labeled ϕ/c . We identify the 3 time-like bivector coefficients with the electric field components E_x/c , E_y/c , E_z/c and the space-like coefficients with the magnetic field components B_x , B_y , B_z .

The components of the 3 space-like bivectors in Eq. (5.4) are the components of the Gibbs' curl \mathbf{A} and may be identified with the magnetic field components

$\mathbf{B} = \text{curl } \mathbf{A}$. Identify $\square \mathbf{A}$ with what is called the electromagnetic field bivector \mathbf{F} and write

$$\begin{aligned}
\Box \mathbf{A} &= \\
\mathbf{F} &= (\mathbf{e}_0 \mathbf{e}_1 E_x + \mathbf{e}_0 \mathbf{e}_2 E_y + \mathbf{e}_0 \mathbf{e}_3 E_z) / c + \mathbf{e}_1 \mathbf{e}_2 B_z + \mathbf{e}_3 \mathbf{e}_1 B_y + \mathbf{e}_2 \mathbf{e}_3 B_x \quad (5.6) \\
&+ \Box \bullet \mathbf{A} \\
\text{or} \\
\mathbf{F} &= (\mathbf{e}_0 \mathbf{e}_1 E_x + \mathbf{e}_0 \mathbf{e}_2 E_y + \mathbf{e}_0 \mathbf{e}_3 E_z) / c + \mathbf{e}_5 (\mathbf{e}_0 \mathbf{e}_1 B_x + \mathbf{e}_0 \mathbf{e}_2 B_y + \mathbf{e}_0 \mathbf{e}_3 B_z) \\
&+ \Box \bullet \mathbf{A} \\
&= \frac{\mathbf{E}}{c} + \mathbf{e}_5 \mathbf{B} + \Box \bullet \mathbf{A} \quad \mathbf{e}_5 = \mathbf{e}_0 \mathbf{e}_1 \mathbf{e}_2 \mathbf{e}_3 \quad \mathbf{e}_5 \mathbf{e}_5 = -1
\end{aligned}$$

where
$$\frac{E_x}{c} = \left(-\frac{\partial A_x}{c \partial t} - \frac{\partial A_0}{\partial x} \right), \quad B_x = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right), \quad \text{etc.}$$

That is,
$$\frac{\mathbf{E}}{c} = -\text{grad} A_0 - \frac{\partial \mathbf{A}'}{c \partial t}, \quad \text{let } A_0 = \frac{\phi}{c}, \quad \mathbf{A}' = \mathbf{e}_1 A_x + \mathbf{e}_2 A_y + \mathbf{e}_3 A_z$$

Then
$$\mathbf{E} = -\text{grad} \phi - \frac{\partial \mathbf{A}'}{\partial t} \quad (5.7)$$

and
$$\mathbf{B} = \text{curl} \mathbf{A}' \quad (5.8)$$

Now write the ‘‘equation of motion’’ of $\Box \mathbf{A}$, that is, of $\mathbf{F} + \Box \bullet \mathbf{A}$ by forming $(\mathbf{F} + \Box \bullet \mathbf{A}) \Box$ and equating it to a physical quantity having the same geometry. A 4-vector source current meets this requirement and, in the spirit of the present treatment, may be regarded as an ansatz to be justified by experiment. Physics enters at this stage. We write the 4-vector source current in two ways. One, the usual, as $\mu_0 \mathbf{J} = \mathbf{e}_0 \mu_0 c^2 \rho + \mu_0 \mathbf{j} = \mathbf{e}_0 \rho / \varepsilon_0 + \mu_0 \mathbf{j}$ and the other for transport of a single charge q_s , $\mu_0 q_s \mathbf{V}_s = \mu_0 q_s (\mathbf{e}_0 c + \mathbf{v}_s)$. ρ = charge density and \mathbf{j} = current density. We choose to write $\mathbf{F} \Box$ rather than $\Box \mathbf{F}$ since the former will be equated to a positive current. Since $\Box \mathbf{F} = -\mathbf{F} \Box$, use of $\Box \mathbf{F}$ would involve a negative current, less aesthetically pleasing.

$$\begin{aligned}
&[\mathbf{F} + \Box \bullet \mathbf{A}] \Box = \\
&\left[\left(\mathbf{e}_0 \mathbf{e}_1 \frac{E_x}{c} + \mathbf{e}_0 \mathbf{e}_2 \frac{E_y}{c} + \mathbf{e}_0 \mathbf{e}_3 \frac{E_z}{c} + \mathbf{e}_1 \mathbf{e}_2 B_z + \mathbf{e}_3 \mathbf{e}_1 B_y + \mathbf{e}_2 \mathbf{e}_3 B_x \right) \right. \\
&\quad \left. + \Box \bullet \mathbf{A} \right] \left(\mathbf{e}_0 \frac{\partial}{c \partial t} + \mathbf{e}_1 \frac{\partial}{\partial x} + \mathbf{e}_2 \frac{\partial}{\partial y} + \mathbf{e}_3 \frac{\partial}{\partial z} \right) \quad (5.9)
\end{aligned}$$

$$\begin{aligned}
 & [\mathbf{F} + \square \bullet \mathbf{A}] \square = \\
 & \mathbf{e}_0 \frac{1}{c} \left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial x} + \frac{\partial E_z}{\partial x} \right) - \mathbf{e}_1 \left(\frac{\partial E_x}{c^2 \partial t} + \frac{\partial B_y}{\partial z} - \frac{\partial B_z}{\partial y} \right) \\
 & - \mathbf{e}_2 \left(\frac{\partial E_y}{c^2 \partial t} + \frac{\partial B_z}{\partial x} - \frac{\partial B_x}{\partial z} \right) - \mathbf{e}_3 \left(\frac{\partial E_z}{c^2 \partial t} + \frac{\partial B_x}{\partial y} - \frac{\partial B_y}{\partial x} \right) \\
 & + \mathbf{e}_0 \mathbf{e}_1 \mathbf{e}_2 \frac{1}{c} \left(-\frac{\partial E_x}{\partial y} + \frac{\partial E_y}{\partial x} - \frac{\partial B_z}{\partial t} \right) + \mathbf{e}_0 \mathbf{e}_1 \mathbf{e}_3 \frac{1}{c} \left(-\frac{\partial E_z}{\partial x} + \frac{\partial E_x}{\partial z} - \frac{\partial B_y}{\partial t} \right) \\
 & + \mathbf{e}_0 \mathbf{e}_2 \mathbf{e}_3 \frac{1}{c} \left(-\frac{\partial E_y}{\partial z} + \frac{\partial E_z}{\partial y} - \frac{\partial B_x}{\partial t} \right) - \mathbf{e}_1 \mathbf{e}_2 \mathbf{e}_3 \left(\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} \right) \\
 & + (\square \bullet \mathbf{A}) \square
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{F} \square &= \mathbf{e}_0 \operatorname{div} \frac{\mathbf{E}}{c} - \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t} + \operatorname{curl} \mathbf{B} + \mathbf{e}_5 \left[-\mathbf{e}_0 \operatorname{div} \mathbf{B} + \frac{\mathbf{e}_1}{c} \left((\operatorname{curl} \mathbf{E})_x + \frac{\partial B_x}{\partial t} \right) \right. \\
 & \quad \left. + \frac{\mathbf{e}_2}{c} \left((\operatorname{curl} \mathbf{E})_y + \frac{\partial B_y}{\partial t} \right) + \frac{\mathbf{e}_3}{c} \left((\operatorname{curl} \mathbf{E})_z + \frac{\partial B_z}{\partial t} \right) \right] \\
 & \quad + (\square \bullet \mathbf{A}) \square \\
 &= \mathbf{e}_0 \operatorname{div} \mathbf{E} - \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t} + \operatorname{curl} \mathbf{B} + \mathbf{e}_5 \left(-\mathbf{e}_0 \operatorname{div} \mathbf{B} + \frac{\partial \mathbf{B}}{\partial t} + \operatorname{curl} \frac{\mathbf{E}}{c} \right) \\
 & \quad + (\square \bullet \mathbf{A}) \square \\
 &= \mu_0 \mathbf{J} = \mathbf{e}_0 \mu_0 c^2 \rho + \mu_0 \mathbf{j} = \mathbf{e}_0 \frac{\rho}{\varepsilon_0} + \mu_0 \mathbf{j} \\
 & \quad \text{or, for a single charge, } q_s \\
 &= \mu_0 q_s \mathbf{V}_s
 \end{aligned} \tag{5.10}$$

Equating coefficients of \mathbf{e}_0 and coefficients of the space vector terms on both sides and then doing the same for the time and space components of \mathbf{e}_5 gives

$$\operatorname{div} \mathbf{E} - \frac{\partial}{c \partial t} (\square \bullet \mathbf{A}) = \frac{\rho}{\varepsilon_0} = \frac{q_s}{\varepsilon_0} \tag{5.11}$$

$$\operatorname{curl} \mathbf{B} - \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} + (\square \bullet \mathbf{A}) \square = \mu_0 \mathbf{j} = \mu_0 q_s \mathbf{v}_s \tag{5.12}$$

$$\operatorname{div} \mathbf{B} = 0 \tag{5.13}$$

$$\operatorname{curl} \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{5.14}$$

Putting $\square \cdot \mathbf{A} \equiv \square \bullet \mathbf{A} = 0$, Eqs. (5.11) and (5.12) are the first two Maxwell's equations. Eqs. (5.13) and (5.14) are the second two Maxwell's equations.

Without $\square \bullet \mathbf{A} = 0$, the first two Maxwell equations do not separate which they must in order to agree with experiment. Therefore we put $\square \bullet \mathbf{A} = 0$, called the Lorentz condition. This is a constraint that is an essential part of Maxwell's equations. Thus after applying the Lorentz condition, Maxwell's equations are:

$$\operatorname{div} \mathbf{E}_s = \frac{\rho_s}{\varepsilon_0} = \frac{q_s}{\varepsilon_0} \quad (5.15)$$

$$\text{and } \mathbf{E}_s + \frac{\partial \mathbf{B}_s}{\partial t} = 0 \quad (5.16)$$

$$\operatorname{div} \mathbf{B}_s = 0 \quad (5.17)$$

$$\operatorname{curl} \mathbf{B}_s - \frac{1}{c^2} \frac{\partial \mathbf{E}_s}{\partial t} = \mu_0 \mathbf{j}_s = \mu_0 q_s \mathbf{v}_s \quad (5.18)$$

We have added a subscript s to \mathbf{E} and \mathbf{B} to indicate explicitly that the field arises from a source current $q_s \mathbf{v}_s$ or \mathbf{j}_s .

If the field quantities \mathbf{E} and \mathbf{B} in Eqs. (5.11) through (5.14) are expressed in terms of their original definition in terms of the derivatives of \mathbf{A} , that is, use $\mathbf{F} = \square \mathbf{A}$, then, after adding and subtracting $\frac{\partial^2 A_0}{c^2 \partial t^2}$ in the coefficient of \mathbf{e}_0 and rearranging, $\mathbf{F} \square$ is:

$$\mathbf{F} \square = \square^2 \mathbf{A} - \square (\square \bullet \mathbf{A})$$

Maxwell's equations do not specify how the fields \mathbf{E} and \mathbf{B} generated by $q_s \mathbf{v}_s$ interact with other charges $q_t \mathbf{v}_t$. The interaction is included by specifying boundary conditions on \mathbf{E}_s and \mathbf{B}_s which implicitly involves the charges $q_t \mathbf{v}_t$ composing the boundary surfaces.

Putting $\square \bullet \mathbf{A} = \frac{\partial A_0}{c \partial t} + \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = 0$, the result is

$$\mathbf{F} \square = \square^2 \mathbf{A} = \mathbf{e}_0 \square^2 A_0 + \mathbf{e}_1 \square^2 A_x + \mathbf{e}_2 \square^2 A_y + \mathbf{e}_3 \square^2 A_z \quad (5.20)$$

Since $\mathbf{F} \square = \mathbf{F} \bullet \square + \mathbf{F} \wedge \square$, we see that $\mathbf{F} \wedge \square$ is identically zero when \mathbf{F} is written in terms of the vector potential as above.

In Eq. (5.20), $\mathbf{F} \square$ is expressed completely in terms of the derivatives of \mathbf{A} .

From Eq. (5.20)

$$\begin{aligned} \square^2 A_0 &= 0, & \square^2 A_x &= 0, & \square^2 A_y &= 0, & \square^2 A_z &= 0 \\ \square^2 &= -\frac{\partial^2}{c^2 \partial t^2} + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \end{aligned}$$

The equations in A_0 , A_x , A_y , and A_z all separate because the Lorentz condition $\square \bullet \mathbf{A} = 0$. (Note that when \square and \mathbf{A} are both vectors, $\square \bullet \mathbf{A} \equiv \square \cdot \mathbf{A}$.)

From Eq. (5.20), including the source quantities on the rhs.

$$\mathbf{e}_0 \square^2 A_0 + \mathbf{e}_2 \square^2 A_x + \mathbf{e}_2 \square^2 A_y + \mathbf{e}_3 \square^2 A_z = \mu_0 \mathbf{J} = \mathbf{e}_0 \frac{q_s}{c\varepsilon_0} + q_s \mathbf{v}_s = \mathbf{e}_0 \frac{\rho}{c\varepsilon_0} + \mu_0 \mathbf{j} \quad (5.21)$$

Thus from the above

$$\begin{aligned} \square^2 A_0 &= \frac{\rho}{c\varepsilon_0}, & \square^2 A_x &= \mu_0 j_x, & \square^2 A_y &= \mu_0 j_y, & \square^2 A_z &= \mu_0 j_z \\ \square^2 &= -\frac{\partial^2}{c^2 \partial t^2} + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \end{aligned}$$

The first term may be written $\square^2 A_0 = \frac{\rho}{c\varepsilon_0} = \frac{q_s}{c\varepsilon_0}$ or $\square^2 \phi = \frac{q_s}{\varepsilon_0} = \frac{\rho}{\varepsilon_0}$ where $A_0 = \frac{\phi}{c}$.

$$\mathbf{F}\square = \mu_0 (c\rho\mathbf{e}_0 + j_x\mathbf{e}_1 + j_y\mathbf{e}_2 + j_z\mathbf{e}_3) = \mu_0 q_s (c\mathbf{e}_0 + v_x\mathbf{e}_1 + v_y\mathbf{e}_2 + v_z\mathbf{e}_3)$$

With $\square \bullet \mathbf{A} = \mathbf{0}$, the dot product, more commonly called the direct product, $\mathbf{F} \bullet \square$, and wedge product, $\mathbf{F} \wedge \square$ yield:

$$\begin{aligned} (\mathbf{F}\square - \square\mathbf{F})/2 &= \mathbf{F} \bullet \square = \mathbf{e}_0 \operatorname{div} \mathbf{E} - \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t} + \operatorname{curl} \mathbf{B} \\ &= \mu_0 \mathbf{J} = \mathbf{e}_0 \mu_0 c^2 \rho + \mu_0 \mathbf{j} = \mathbf{e}_0 \frac{\rho}{\varepsilon_0} + \mu_0 \mathbf{j} = \mu_0 q_s (\mathbf{e}_0 c + \mathbf{v}_s) \end{aligned} \quad (5.22)$$

$$(\mathbf{F}\square + \square\mathbf{F})/2 = \mathbf{F} \wedge \square = \mathbf{e}_5 \left(\mathbf{e}_0 \operatorname{div} \mathbf{B} - \frac{\partial \mathbf{B}}{\partial t} - \operatorname{curl} \mathbf{E} \right) = \mathbf{e}_5 \mu_0 q \mathbf{U} \quad (5.23)$$

\mathbf{F} , Eq. (5.6), may be arranged, where the sum of each half is \mathbf{F} .

$$\mathbf{F} = \begin{vmatrix} & +\mathbf{e}_0\mathbf{e}_1 E_x/c & +\mathbf{e}_0\mathbf{e}_2 E_y/c & +\mathbf{e}_0\mathbf{e}_3 E_z/c \\ -\mathbf{e}_1\mathbf{e}_0 E_x/c & & +\mathbf{e}_1\mathbf{e}_2 B_z & +\mathbf{e}_3\mathbf{e}_1 B_y \\ -\mathbf{e}_2\mathbf{e}_0 E_y/c & -\mathbf{e}_2\mathbf{e}_1 B_z & & +\mathbf{e}_2\mathbf{e}_3 B_x \\ -\mathbf{e}_3\mathbf{e}_0 E_z/c & -\mathbf{e}_3\mathbf{e}_1 B_y & -\mathbf{e}_3\mathbf{e}_2 B_x & \end{vmatrix} \quad (5.24)$$

the ordering of the unit vectors labels the elements of a tensor. Removing the unit vectors displays the terms as a tensor in the usual way

$$\mathbf{F} = \begin{vmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & -B_y & -B_x & 0 \end{vmatrix} \quad (5.25)$$

The coefficients constitute the antisymmetric field strength tensor. Calling the components $F^{\mu\nu}$, the first half of Maxwell's equations, Eqs. (5.15), (5.16) are traditionally written

$$\partial_\mu F^{\mu\nu} = \mu_0 j^\nu, \quad \text{where} \quad j^0 = c\rho, \quad j^1 = j_x, \quad j^2 = j_y, \quad j^3 = j_z. \quad (5.26)$$

Evaluating for $\nu = 0$, for example, we obtain

$$\frac{\partial(0)}{c\partial t} + \frac{\partial E_x}{c\partial x} + \frac{\partial E_y}{c\partial y} + \frac{\partial E_z}{c\partial z} = \mu_0 c\rho, \quad \text{or} \quad \text{div}\mathbf{E} = \mu_0 c^2 \rho = \rho/\epsilon_0 = \frac{q_s}{\epsilon_0}$$

The homogeneous pair of Maxwell's equations, are usually summarized by

$$\partial_\mu F^{\nu\lambda} + \partial_\nu F^{\lambda\mu} + \partial_\lambda F^{\mu\nu} = 0 \quad (5.27)$$

where μ, ν, λ are any three of the integers 0, 1, 2, 3, namely 0 1 2, 0 1 3, 0 2 3 and 1 2 3, corresponding to the coefficients of the trivectors in Eq. (4). Thus, for $\mu = 0, \nu = 1, \lambda = 2$

$$\frac{\partial F^{12}}{c\partial t} + \frac{\partial F^{20}}{\partial x} + \frac{\partial F^{01}}{\partial y} = 0 \quad \text{that is,} \quad \frac{\partial E_z}{c\partial t} + \frac{\partial E_y}{c\partial x} + \frac{\partial E_x}{c\partial y} = 0, \quad \text{etc.}$$

Written in space time algebra, Maxwell's equations are simply given by

$$\mathbf{F}\square = \mu_0 \mathbf{J} \quad (5.28)$$

or

$$\square\mathbf{F} = -\mu_0 \mathbf{J}$$

with $\square \bullet \mathbf{A} = \mathbf{A} \bullet \square = 0$.

Eqs. (5.26) and (5.27) are obviously awkward ways of summarizing Maxwell's equations. They are not conveniently amenable to further manipulation, as is Eq. (5.28).

5.1.1 Summary

Apply $\widehat{\square}$ to \mathbf{A} to obtain

$$\mathbf{A}\square = \mathbf{A} \wedge \square + \mathbf{A} \bullet \square \quad (5.29)$$

We put the last term equal to zero and call $\mathbf{A} \wedge \square$, \mathbf{F} . Thus,

$$\mathbf{A} \wedge \square = \mathbf{F} \quad (5.30)$$

We regard \mathbf{F} as a bivector in the field quantities \mathbf{E} and \mathbf{B} .

Now form

$$\mathbf{F}\square = (\mathbf{F} \bullet \square + \mathbf{F} \wedge \square) \quad (5.31)$$

where

$$\mathbf{F} \bullet \square = (\mathbf{F}\square + \square\mathbf{F})/2, \quad \mathbf{F} \wedge \square = (\mathbf{F}\square - \square\mathbf{F})/2$$

$\mathbf{F} \bullet \square$ is a vector, which we equate to a four-current $\mu_0\mathbf{J}$. Since \mathbf{F} is derived from a vector, \mathbf{A} , the second term in Eq. (5.31), $\mathbf{F} \wedge \square = \mathbf{A} \wedge \square \wedge \square$, is identically zero.

$$\mathbf{F} \bullet \square = \mu_0\mathbf{J} \quad (5.32)$$

$$\mathbf{F} \wedge \square = 0 \quad (5.33)$$

These are Maxwell's equations, or simply,

$$\mathbf{F}\square = \mu\mathbf{J} \quad (5.34)$$

Now we express the results in terms of \mathbf{A} as follows

$$\mathbf{F} \bullet \square = \mu_0\mathbf{J}$$

where $\mathbf{F} = \mathbf{A} \wedge \square$. Therefore,

$$\begin{aligned} \mathbf{F} \bullet \square &= \frac{(\mathbf{A}\square - \square\mathbf{A})}{2} \bullet \square = \frac{1}{4} [(\mathbf{A}\square - \square\mathbf{A})\square + \square(\mathbf{A}\square - \square\mathbf{A})] \\ &= \frac{\mathbf{A}\square\square}{2} - \frac{\square\square\mathbf{A}}{2} = \square^2\mathbf{A} \end{aligned}$$

Therefore

$$\square^2\mathbf{A} = -\mu\mathbf{J} = \mu_0q_s\mathbf{v}_s \quad (5.35)$$

which expresses Maxwell's equations in terms of \mathbf{A} .

Note that if we had started with a conserved current condition

$$\square \bullet \mathbf{J} = 0 \quad (5.36)$$

we could have worked backwards from the preceding steps and deduced the form of the electromagnetic field, the Maxwell equations, from current conservation. That is, proceed as follows.

We know that \square applied to a bivector \mathbf{F} yields a bivector plus a trivector

$$\mathbf{F}\square = \mathbf{F} \bullet \square + \mathbf{F} \wedge \square = \text{vector} + \text{trivector}$$

If we put $\mathbf{F} \wedge \square = 0$ and identify $\mathbf{F} \bullet \square$ with $\mu_0 \mathbf{J}$, then Eq. (5.36) is satisfied. Thus

$$\square \bullet \square \bullet \mathbf{F} \equiv 0 \quad \text{or} \quad \mathbf{F} \bullet \square \bullet \square \equiv 0 \quad (5.37)$$

which is satisfied identically for *any* multivector. It is also true that $\square \wedge \square \wedge \mathbf{F}$ or $\mathbf{F} \wedge \square \wedge \square \equiv 0$.

From Eq. (5.37) Maxwell's equations are

$$\begin{aligned} \mathbf{F} \bullet \square &= \mu_0 \mathbf{J} = \mathbf{e}_0 \mu_0 c^2 \rho + \mu_0 \mathbf{j} = \mathbf{e}_0 \frac{\rho}{\varepsilon_0} + \mu_0 \mathbf{j} \\ &= \mathbf{e}_0 \frac{q_s}{\varepsilon_0} + \mu_0 q_s \mathbf{v}_s \end{aligned} \quad (5.38)$$

$$\text{and} \quad \mathbf{F} \wedge \square = 0 \quad (5.39)$$

$$\text{or simply} \quad \mathbf{F} \square = \mu_0 \mathbf{J} \quad (5.40)$$

If \mathbf{F} is derivable from a potential through

$$\square \mathbf{A} = \square \wedge \mathbf{A} + \square \bullet \mathbf{A}$$

and if we put $\square \bullet \mathbf{A} = 0$, then Eq. (15b) is satisfied identically since

$$\square \wedge \square \wedge \mathbf{A} \equiv 0$$

By this argument, Maxwell's equations are a result of current conservation.

It is interesting to form

$$\begin{aligned} \mathbf{F}_t \mathbf{F}_s &= \left(\mathbf{B}_t \cdot \mathbf{B}_s - \frac{\mathbf{E}_t \cdot \mathbf{E}_s}{c} \right) + \mathbf{e}_5 \left(\frac{\mathbf{E}_t \cdot \mathbf{B}_s + \mathbf{B}_t \cdot \mathbf{E}_s}{c} \right) \\ &+ \mathbf{e}_0 \mathbf{e}_1 \left[\left(\frac{\mathbf{E}_s}{c} \times \mathbf{B}_t \right)_x - \left(\frac{\mathbf{E}_t}{c} \times \mathbf{B}_s \right)_x \right] \\ &+ \mathbf{e}_0 \mathbf{e}_2 \left[- \left(\frac{\mathbf{E}_t}{c} \times \mathbf{B}_s \right)_y + \left(\frac{\mathbf{E}_s}{c} \times \mathbf{B}_t \right)_y \right] \\ &+ \mathbf{e}_0 \mathbf{e}_3 \left[\left(\frac{\mathbf{E}_s}{c} \times \mathbf{B}_t \right)_z - \left(\frac{\mathbf{E}_t}{c} \times \mathbf{B}_s \right)_z \right] \\ &+ \mathbf{e}_1 \mathbf{e}_2 \left[\left(\frac{\mathbf{E}_t}{c} \times \frac{\mathbf{E}_s}{c} \right)_z - (\mathbf{B}_t \times \mathbf{B}_s)_z \right] \\ &+ \mathbf{e}_3 \mathbf{e}_1 \left[\left(\frac{\mathbf{E}_t}{c} \times \frac{\mathbf{E}_s}{c} \right)_y - (\mathbf{B}_t \times \mathbf{B}_s)_y \right] \\ &+ \mathbf{e}_2 \mathbf{e}_3 \left[\left(\frac{\mathbf{E}_t}{c} \times \frac{\mathbf{E}_s}{c} \right)_x - (\mathbf{B}_t \times \mathbf{B}_s)_x \right] \end{aligned}$$

The bivector coefficients are all zero in this case when $\mathbf{F}_t = \mathbf{F}_s \rightarrow \mathbf{F}$.

$$\mathbf{F}\mathbf{F} = \mathbf{B}^2 - \frac{\mathbf{E}^2}{c^2} + \mathbf{e}_5 \left(\frac{\mathbf{E}}{c} \cdot \mathbf{B} \right) = 0$$

The above two terms are scalars and therefore are Lorentz invariants, that is, true scalars, no unit vectors involved.

$$\mathbf{B}^2 - \frac{\mathbf{E}^2}{c^2} = \text{inv} = \mathbf{B}'^2 - \frac{\mathbf{E}'^2}{c^2} \quad (5.41)$$

$$\frac{\mathbf{E} \cdot \mathbf{B}}{c} = \frac{\mathbf{E}'}{c} \cdot \mathbf{B}' \quad (5.42)$$

In problem 14-10, Heald and Marion (1995, 529) say, “Use these results to show that the quantity

$$(\mathbf{E} \times \mathbf{B})^2 - c^2 (\mathbf{E}^2 + \mathbf{B}^2)$$

is invariant, where $(\mathbf{E} \times \mathbf{B})^2$ is the Poynting magnitude and $c^2 (\mathbf{E}^2 + \mathbf{B}^2)$ is the energy density of the fields.”

5.2 Comments on the Source Current J

We define a space-time vector as a covariant vector but whose square is not an invariant. An absolute invariant is one whose square is a scalar and therefore has the same value in all inertial frames. As an example of each of the above, consider a velocity vector given by \mathbf{V} and \mathbf{V}' respectively in two Lorentz frames.

$$\begin{aligned} \mathbf{V} &= \mathbf{e}_0 c + \mathbf{e}_1 v_x + \mathbf{e}_2 v_y + \mathbf{e}_3 v_z \\ \mathbf{V}' &= \mathbf{e}_0 c + \mathbf{e}_1 v'_x + \mathbf{e}_2 v'_y + \mathbf{e}_3 v'_z \\ \text{where} \quad \mathbf{V}' &= \mathbf{e}^{-\mathbf{e}_1 \mathbf{e}_2 \theta/2} \mathbf{V} \mathbf{e}^{\mathbf{e}_1 \mathbf{e}_2 \theta/2} \end{aligned}$$

\mathbf{V} is covariant since it has the same form under a Lorentz transformation. Note that all quantities that are expressed in terms of $\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ or their products are automatically covariant. Thus Maxwell’s equations are covariant, and nothing further need be done to demonstrate that. The square of \mathbf{V}

$$V^2 = -c^2 + v^2 = V'^2 = -c^2 + v'^2$$

is not an absolute invariant since the most one can say is that $\mathbf{v}^2 = \mathbf{v}'^2$. That is, both quantities are equal but may have arbitrary values. On the other hand, the velocity vector

$$\mathbf{V} = \gamma (\mathbf{e}_0 c + \mathbf{e}_1 v_x + \mathbf{e}_2 v_y + \mathbf{e}_3 v_z) \quad \gamma = 1/\sqrt{1 - v^2/c^2}$$

is a 4-velocity. Its square is an absolute invariant, $-c^2$.

$$\mathbf{V} \cdot \mathbf{V} = \gamma^2 (-c^2 + v^2) = -c^2$$

A space-time current for a single charge is

$$\begin{aligned} \mathbf{J} &= q_s (\mathbf{e}_0 c + \mathbf{v}) \\ \mathbf{J}^2 &= q_s^2 (-c^2 + v^2) = \rho_0^2 c^2 + j^2 \end{aligned}$$

$$\text{while } \mathbf{J} = q\mathbf{V} = \gamma q (\mathbf{e}_0 c + \mathbf{v}) = \gamma (\mathbf{e}_0 \rho_0 c + \rho_0 \mathbf{v}) = \gamma (\mathbf{e}_0 \rho_0 c + \mathbf{j}_0)$$

is a 4-current. q_s is charge, an invariant, ρ_0 is rest charge density and

$$\begin{aligned} \mathbf{J}^2 &= \gamma^2 q_s^2 (-c^2 + v^2) = -q_s^2 c^2 \\ \text{or } \mathbf{J}^2 &= \gamma^2 (-\rho_0^2 c^2 + \rho_0^2 v^2) = \gamma^2 \rho_0^2 (-c^2 + v^2) = -c^2 \rho_0^2 \end{aligned}$$

5.3 Derivation of Maxwell's Equations in a Schwarzschild Metric

We sketch a derivation of Maxwell's equations in a Schwarzschild metric by following the procedure adopted for the Cartesian metric. That is, we form $\square \mathbf{A}$ followed by $\square(\square \mathbf{A})$ and equate the result to a source current. The difference is that spherical coordinates with a curvature factor $n(r)$ are employed. The results are applicable to spherical coordinates alone by putting $n(r) = 1$.

5.4 Schwarzschild Metric

$$\begin{aligned} ds &= \mathbf{e}_0 c h_t dt + \mathbf{e}_r h_r dr + \mathbf{e}_\theta h_\theta d\theta + \mathbf{e}_\varphi h_\varphi d\varphi \\ ds &= \mathbf{e}_0 c (1 - 2\alpha/r)^{1/2} dt + \frac{\mathbf{e}_r dr}{(1 - 2\alpha/r)^{1/2}} + \mathbf{e}_\theta r d\theta + \mathbf{e}_\varphi r \sin \theta d\varphi \end{aligned} \quad (5.43)$$

$$\square = \mathbf{e}_0 \frac{1}{(1 - 2\alpha/r)^{1/2}} \frac{\partial}{c \partial t} + \mathbf{e}_r (1 - 2\alpha/r)^{1/2} \frac{\partial}{\partial r} + \mathbf{e}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + \mathbf{e}_\varphi \frac{1}{r \sin \theta} \frac{\partial}{\partial \varphi} \quad (5.44)$$

$$\text{Let } \frac{1}{(1 - 2\alpha/r)^{1/2}} = n(r), \quad (1 - 2\alpha/r)^{1/2} = 1/n(r), \quad \alpha = \frac{GM}{c^2} \quad (5.45)$$

$$A = \mathbf{e}_0 A_0 + \mathbf{e}_r A_r + \mathbf{e}_\theta A_\theta + \mathbf{e}_\varphi A_\varphi \quad (5.46)$$

Derivatives of the unit vectors are:

$$\begin{aligned}
\frac{\partial \mathbf{e}_r}{\partial \theta} &= \mathbf{e}_\theta & \frac{\partial \mathbf{e}_r}{\partial \varphi} &= \mathbf{e}_\varphi \sin \theta & \frac{\partial \mathbf{e}_r}{\partial r} &= 0 \\
\frac{\partial \mathbf{e}_\theta}{\partial \theta} &= -\mathbf{e}_r & \frac{\partial \mathbf{e}_\theta}{\partial \varphi} &= \mathbf{e}_\varphi \cos \theta & \frac{\partial \mathbf{e}_\theta}{\partial r} &= 0 \\
\frac{\partial \mathbf{e}_\varphi}{\partial \theta} &= 0 & \frac{\partial \mathbf{e}_\varphi}{\partial \varphi} &= -\mathbf{e}_\theta \cos \theta - \mathbf{e}_r \sin \theta & \frac{\partial \mathbf{e}_\varphi}{\partial r} &= 0
\end{aligned} \tag{5.47}$$

$$\begin{aligned}
\frac{\partial}{\partial \theta} \mathbf{e}_r \mathbf{e}_r &= 0 & \frac{\partial}{\partial \varphi} \mathbf{e}_r \mathbf{e}_r &= 0 & \frac{\partial}{\partial r} \mathbf{e}_r \mathbf{e}_r &= 2\mathbf{e}_r^2 \frac{\partial \mathbf{e}_r}{\partial r} = 0 \\
\frac{\partial}{\partial \theta} \mathbf{e}_r \mathbf{e}_\theta &= 0 & \frac{\partial}{\partial \varphi} \mathbf{e}_r \mathbf{e}_\theta &= \mathbf{e}_\varphi \mathbf{e}_\theta \sin \theta + \mathbf{e}_r \mathbf{e}_\varphi \cos \theta & \frac{\partial}{\partial r} \mathbf{e}_r \mathbf{e}_\theta &= \mathbf{e}_r \frac{\partial \mathbf{e}_\theta}{\partial r} = 0 \\
\frac{\partial}{\partial \theta} \mathbf{e}_r \mathbf{e}_\varphi &= \mathbf{e}_\theta \mathbf{e}_\varphi & \frac{\partial}{\partial \varphi} \mathbf{e}_r \mathbf{e}_\varphi &= \mathbf{e}_\theta \mathbf{e}_\varphi & \frac{\partial}{\partial r} \mathbf{e}_r \mathbf{e}_\varphi &= \mathbf{e}_r \frac{\partial \mathbf{e}_\varphi}{\partial r} = 0
\end{aligned} \tag{5.48}$$

The electromagnetic trivector in spherical coordinates is:

$$\mathbf{e}_0 \mathbf{e}_r \frac{E_r}{c} + \mathbf{e}_0 \mathbf{e}_\theta \frac{E_\theta}{c} + \mathbf{e}_0 \mathbf{e}_\varphi \frac{E_\varphi}{c} + \mathbf{e}_r \mathbf{e}_\theta B_\varphi + \mathbf{e}_\varphi \mathbf{e}_r B_\theta + \mathbf{e}_\theta \mathbf{e}_\varphi B_r$$

The following derivatives of the products of unit vectors are required.

$$\begin{aligned}
\frac{\partial}{\partial \theta} \mathbf{e}_r \mathbf{e}_r &= \mathbf{e}_r \frac{\partial \mathbf{e}_r}{\partial \theta} + \frac{\partial \mathbf{e}_r}{\partial \theta} \mathbf{e}_r & \frac{\partial}{\partial \varphi} \mathbf{e}_r \mathbf{e}_r &= \mathbf{e}_r \mathbf{e}_\varphi \sin \theta + \mathbf{e}_\varphi \cos \theta \mathbf{e}_r \\
&= \mathbf{e}_r \mathbf{e}_\theta + \mathbf{e}_\theta \mathbf{e}_r = 0 & &= \sin \theta (\mathbf{e}_r \mathbf{e}_\varphi + \mathbf{e}_\varphi \mathbf{e}_r) = 0 \\
\frac{\partial}{\partial \theta} \mathbf{e}_r \mathbf{e}_\theta &= \frac{\partial \mathbf{e}_r}{\partial \theta} \mathbf{e}_\theta + \mathbf{e}_r \frac{\partial \mathbf{e}_\theta}{\partial \theta} & \frac{\partial}{\partial \varphi} \mathbf{e}_r \mathbf{e}_\theta &= \frac{\partial \mathbf{e}_r}{\partial \varphi} \mathbf{e}_\theta + \mathbf{e}_r \frac{\partial \mathbf{e}_\theta}{\partial \varphi} \\
&= \mathbf{e}_\theta \mathbf{e}_\theta - \mathbf{e}_r \mathbf{e}_r = 0 & &= \mathbf{e}_\varphi \mathbf{e}_\theta \sin \theta + \mathbf{e}_r \mathbf{e}_\varphi \cos \theta \\
\frac{\partial}{\partial \theta} \mathbf{e}_r \mathbf{e}_\varphi &= \mathbf{e}_\theta \mathbf{e}_\varphi + \mathbf{e}_r \cdot 0 & \frac{\partial}{\partial \varphi} \mathbf{e}_r \mathbf{e}_\varphi &= \mathbf{e}_\varphi \sin \theta \mathbf{e}_\varphi + \mathbf{e}_r \frac{\partial \mathbf{e}_\varphi}{\partial \varphi} \\
&= \mathbf{e}_\theta \mathbf{e}_\varphi & &= \sin \theta + \mathbf{e}_r (-\mathbf{e}_\theta \cos \theta - \mathbf{e}_r \sin \theta) \\
& & &= -\mathbf{e}_r \mathbf{e}_\theta \cos \theta
\end{aligned}$$

$A_0, A_r, A_\theta, A_\varphi$ are given by:

$$\begin{array}{cccc}
A_0 & A_r & A_\theta & A_\varphi \\
\mathbf{e}_0 n(r) c dt & \mathbf{e}_r \frac{1}{n(r)} \partial r & \mathbf{e}_\theta r \partial \theta & \mathbf{e}_\varphi r \sin \theta \partial \varphi
\end{array} \tag{5.49}$$

$$\begin{aligned}
\Box \mathbf{A} &= \\
&\left[-\mathbf{e}_0 n(r) \frac{\partial}{c \partial t} + \mathbf{e}_r \frac{1}{n(r)} \frac{\partial}{\partial r} + \mathbf{e}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + \mathbf{e}_\varphi \frac{1}{r \sin \theta} \frac{\partial}{\partial \varphi} \right] \\
&[\mathbf{e}_0 A_0 + \mathbf{e}_r A_r + \mathbf{e}_\theta A_\theta + \mathbf{e}_\varphi A_\varphi] \\
= &-\mathbf{e}_0 n(r) \frac{1}{c} \frac{\partial}{\partial t} (\mathbf{e}_0 A_0) - \mathbf{e}_0 n(r) \frac{1}{c} \frac{\partial}{\partial t} (\mathbf{e}_r A_r) - \mathbf{e}_0 n(r) \frac{1}{c} \frac{\partial}{\partial t} (\mathbf{e}_\theta A_\theta) \\
&-\mathbf{e}_0 n(r) \frac{1}{c} \frac{\partial}{\partial t} (\mathbf{e}_\varphi A_\varphi) \\
&+\mathbf{e}_r \frac{1}{n(r)} \frac{\partial}{\partial r} (\mathbf{e}_0 A_0) + \mathbf{e}_r \frac{1}{n(r)} \frac{\partial}{\partial r} (\mathbf{e}_r A_r) + \mathbf{e}_r \frac{1}{n(r)} \frac{\partial}{\partial r} (\mathbf{e}_\theta A_\theta) \\
&+\mathbf{e}_r \frac{1}{n(r)} \frac{\partial}{\partial r} (\mathbf{e}_\varphi A_\varphi) \\
&+\mathbf{e}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} (\mathbf{e}_0 A_0) + \mathbf{e}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} (\mathbf{e}_r A_r) + \mathbf{e}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} (\mathbf{e}_\theta A_\theta) \\
&+\mathbf{e}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} (\mathbf{e}_\varphi A_\varphi) \\
&+\mathbf{e}_\varphi \frac{1}{r \sin \theta} \frac{\partial}{\partial \varphi} (\mathbf{e}_0 A_0) + \mathbf{e}_\varphi \frac{1}{r \sin \theta} \frac{\partial}{\partial \varphi} (\mathbf{e}_r A_r) + \mathbf{e}_\varphi \frac{1}{r \sin \theta} \frac{\partial}{\partial \varphi} (\mathbf{e}_\theta A_\theta) \\
&+\mathbf{e}_\varphi \frac{1}{r \sin \theta} \frac{\partial}{\partial \varphi} (\mathbf{e}_\varphi A_\varphi) \tag{5.50}
\end{aligned}$$

First differentiate A_0 , A_r , A_θ and A_φ .

$$\begin{aligned}
\Box \mathbf{A} &= -\mathbf{e}_0 \mathbf{e}_0 n(r) \frac{\partial A_0}{c \partial t} - \mathbf{e}_0 \mathbf{e}_r n(r) \frac{\partial A_r}{c \partial t} - \mathbf{e}_0 \mathbf{e}_\theta n(r) \frac{\partial A_\theta}{c \partial t} - \mathbf{e}_0 \mathbf{e}_\varphi n(r) \frac{\partial A_\varphi}{c \partial t} \\
&+ \mathbf{e}_r \mathbf{e}_0 \frac{1}{n(r)} \frac{\partial A_0}{\partial r} + \mathbf{e}_r \frac{1}{n(r)} \left(\mathbf{e}_\theta A_r + \mathbf{e}_r \frac{\partial A_r}{\partial r} \right) + \mathbf{e}_r \frac{1}{n(r)} \mathbf{e}_\theta \frac{\partial A_\theta}{\partial r} \\
&+ \mathbf{e}_r \frac{1}{n(r)} \mathbf{e}_\varphi \frac{\partial A_\varphi}{\partial r} \\
&+ \mathbf{e}_\theta \mathbf{e}_0 \frac{1}{r} \frac{\partial A_0}{\partial \theta} + \mathbf{e}_\theta \frac{1}{r} \left(\mathbf{e}_\theta A_r + \mathbf{e}_r \frac{\partial A_r}{\partial \theta} \right) + \mathbf{e}_\theta \frac{1}{r} \left(-\mathbf{e}_r A_\theta + \mathbf{e}_\theta \frac{\partial A_\theta}{\partial r} \right) \\
&+ \mathbf{e}_\theta \frac{1}{r} \left(\mathbf{e}_\varphi \frac{\partial A_\varphi}{\partial r} \right) \\
&+ \mathbf{e}_\varphi \mathbf{e}_0 \frac{1}{r \sin \theta} \frac{\partial A_0}{\partial \varphi} + \mathbf{e}_\varphi \frac{1}{r \sin \theta} \left(\mathbf{e}_\varphi \sin \theta A_r + \mathbf{e}_r \frac{\partial A_r}{\partial \varphi} \right) \\
&+ \mathbf{e}_\varphi \frac{1}{r \sin \theta} \left(\mathbf{e}_\varphi \cos \theta A_\theta + \mathbf{e}_\theta \frac{\partial A_\theta}{\partial \varphi} \right) \\
&+ \mathbf{e}_\varphi \frac{1}{r \sin \theta} \left[(-\mathbf{e}_\theta \cos \theta - \mathbf{e}_r \sin \theta) A_\varphi + \mathbf{e}_\varphi \frac{\partial A_\varphi}{\partial \varphi} \right]
\end{aligned}$$

The bivector terms are

$$\begin{aligned}
&= -\mathbf{e}_0 \mathbf{e}_0 n(r) \frac{\partial A_0}{c \partial t} - \mathbf{e}_0 \mathbf{e}_r n(r) \frac{\partial A_r}{c \partial t} - \mathbf{e}_0 \mathbf{e}_\theta n(r) \frac{\partial A_\theta}{c \partial t} - \mathbf{e}_0 \mathbf{e}_\varphi n(r) \frac{\partial A_\varphi}{c \partial t} \\
&+ \mathbf{e}_r \mathbf{e}_0 \frac{1}{n(r)} \frac{\partial A_0}{\partial r} + \mathbf{e}_r \mathbf{e}_r \frac{1}{n(r)} \frac{\partial A_r}{\partial r} + \mathbf{e}_r \mathbf{e}_\theta \frac{1}{n(r)} \left(\frac{\partial A_\theta}{\partial r} + A_r \right) + \mathbf{e}_r \mathbf{e}_\varphi \frac{\partial A_\varphi}{\partial r} \\
&+ \mathbf{e}_\theta \mathbf{e}_0 \frac{1}{r} \frac{\partial A_0}{\partial \theta} + \mathbf{e}_\theta \mathbf{e}_r \frac{1}{r} \left(\frac{\partial A_r}{\partial \theta} - A_\theta \right) + \mathbf{e}_\theta \mathbf{e}_\theta \frac{1}{r} \left(\frac{\partial A_\theta}{\partial r} + A_r \right) + \mathbf{e}_\theta \mathbf{e}_\varphi \frac{1}{r} \frac{\partial A_\varphi}{\partial \theta} \\
&+ \mathbf{e}_\varphi \mathbf{e}_0 \frac{1}{r \sin \theta} \frac{\partial A_0}{\partial \varphi} + \mathbf{e}_\varphi \mathbf{e}_r \frac{1}{r \sin \theta} \left(\frac{\partial A_r}{\partial \varphi} - A_\varphi \sin \theta \right) \\
&+ \mathbf{e}_\varphi \mathbf{e}_\theta \frac{1}{r \sin \theta} \left(\frac{\partial A_\theta}{\partial \varphi} - A_\varphi \cos \theta \right) \\
&+ \mathbf{e}_\varphi \mathbf{e}_\varphi \frac{1}{r \sin \theta} \left(\frac{\partial A_\varphi}{\partial \varphi} + \sin \theta A_r + \frac{\cos \theta}{\sin \theta} A_\theta \right) \tag{5.51}
\end{aligned}$$

Collecting diagonal terms (scalar)

$$n(r) \frac{\partial A_0}{c \partial t} + \frac{1}{n(r)} \frac{\partial A_r}{\partial r} + \frac{1}{r} \left(\frac{\partial A_\theta}{\partial r} + A_r \right) + \frac{1}{r \sin \theta} \left(\frac{\partial A_\varphi}{\partial \varphi} + \sin \theta A_r + \frac{\cos \theta}{\sin \theta} A_\varphi \right) \tag{5.52}$$

Equating the diagonal terms to zero is the Lorentz condition.

The coefficients of the bivector terms are (analogue of $\left[\mathbf{e}_0\mathbf{e}_x\frac{E_x}{c} + \dots\right]$)

$$\begin{aligned}
\Box\mathbf{A}=\mathbf{F} &= \mathbf{e}_0\mathbf{e}_r \left(-n(r)\frac{\frac{E_r}{c}}{c\partial t} - \frac{1}{n(r)}\frac{\partial A_0}{\partial r} \right) + \mathbf{e}_0\mathbf{e}_\theta \left(-n(r)\frac{\frac{E_\theta}{c}}{c\partial t} - \frac{1}{r}\frac{\partial A_\theta}{\partial r} \right) \\
&+ \mathbf{e}_0\mathbf{e}_\varphi \left(-n(r)\frac{\frac{E_\varphi}{c}}{c\partial t} - \frac{1}{r}\frac{\partial A_\theta}{\partial r} \right) \\
&+ \mathbf{e}_r\mathbf{e}_\theta \left[\frac{1}{n(r)}\left(\frac{\partial A_\theta}{\partial r} + A_r\right) - \frac{1}{r}\left(\frac{\partial A_r}{\partial \theta} - A_\theta\right) \right] \\
&+ \mathbf{e}_\varphi\mathbf{e}_r \left[\frac{1}{r\sin\theta}\left(\frac{\partial A_r}{\partial \varphi} - A_\varphi\sin\theta - \frac{\partial A_\varphi}{\partial r}\right) \right] \\
&+ \mathbf{e}_\theta\mathbf{e}_\varphi \left[\frac{1}{r}\frac{\partial A_\varphi}{\partial r} - \frac{1}{r\sin\theta}\left(\frac{\partial A_\theta}{\partial \varphi} - A_\varphi\cos\theta\right) \right] \tag{5.53}
\end{aligned}$$

Re-labeling coefficients

$$\begin{aligned}
\mathbf{F} &= \mathbf{e}_0\mathbf{e}_r\frac{E_r}{c} + \mathbf{e}_0\mathbf{e}_\theta\frac{E_\theta}{c} + \mathbf{e}_0\mathbf{e}_\varphi\frac{E_\varphi}{c} + \mathbf{e}_r\mathbf{e}_\theta B_\varphi + \mathbf{e}_\varphi\mathbf{e}_r B_\theta + \mathbf{e}_\theta\mathbf{e}_\varphi B_r \tag{5.54} \\
\frac{E_r}{c} &= -n(r)\frac{\partial A_r}{c\partial t} - \frac{1}{n(r)}\frac{\partial A_0}{\partial r} & B_{r\theta} &= \frac{1}{n(r)}\left(\frac{\partial A_\theta}{\partial r} + A_r\right) - \frac{1}{r}\left(\frac{\partial A_r}{\partial \theta} - A_\theta\right) \\
\frac{E_\theta}{c} &= -n(r)\frac{\partial A_\theta}{c\partial t} - \frac{1}{r}\frac{\partial A_\theta}{\partial r} & B_{\varphi r} &= \frac{1}{r\sin\theta}\left(\frac{\partial A_r}{\partial \varphi} - A_\varphi\sin\theta - \frac{\partial A_\varphi}{\partial r}\right) \\
\frac{E_\varphi}{c} &= -n(r)\frac{\partial A_\varphi}{c\partial t} - \frac{1}{r}\frac{\partial A_\theta}{\partial r} & B_{\theta\varphi} &= \frac{1}{r}\frac{\partial A_\varphi}{\partial r} - \frac{1}{r\sin\theta}\left(\frac{\partial A_\theta}{\partial \varphi} - A_\varphi\cos\theta\right)
\end{aligned}$$

Now form $\Box\mathbf{F}$, differentiating both the coefficients and the unit vectors with respect to the coordinates. $\frac{\partial}{\partial t}(\text{unit vectors}) = 0$.

$$\begin{aligned}
\Box\mathbf{F} &= \left(-\mathbf{e}_0n(r)\frac{\partial}{c\partial t} + \mathbf{e}_r\frac{1}{n(r)}\frac{\partial}{\partial r} + \mathbf{e}_\theta\frac{1}{r}\frac{\partial}{\partial \theta} + \mathbf{e}_\varphi\frac{1}{r\sin\theta}\frac{\partial}{\partial \varphi} \right) \times \\
&\left(\mathbf{e}_0\mathbf{e}_r\frac{E_r}{c} + \mathbf{e}_0\mathbf{e}_\theta\frac{E_\theta}{c} + \mathbf{e}_0\mathbf{e}_\varphi\frac{E_\varphi}{c} + \mathbf{e}_r\mathbf{e}_\theta B_\varphi + \mathbf{e}_\varphi\mathbf{e}_r B_\theta + \mathbf{e}_\theta\mathbf{e}_\varphi B_r \right)
\end{aligned}$$

$$\begin{aligned}
&= \mathbf{e}_r n(r) \frac{\partial E_r}{c^2 \partial t} + \mathbf{e}_\theta n(r) \frac{1}{c^2} \frac{\partial E_\theta}{\partial t} + \mathbf{e}_\varphi n(r) \frac{\partial E_\varphi}{c^2 \partial t} \\
&\quad - \mathbf{e}_0 \mathbf{e}_r \mathbf{e}_\theta n(r) \frac{\partial B_\varphi}{c \partial t} - \mathbf{e}_0 \mathbf{e}_\varphi \mathbf{e}_r n(r) \frac{\partial B_\theta}{c \partial t} - \mathbf{e}_0 \mathbf{e}_\theta \mathbf{e}_\varphi n(r) \frac{\partial B_r}{c \partial t} \\
&\quad \text{-----} \\
&\quad + \mathbf{e}_r \mathbf{e}_0 \mathbf{e}_r \frac{1}{n(r)} \frac{\partial E_r}{c \partial r} + \mathbf{e}_r \mathbf{e}_0 \mathbf{e}_\theta \frac{1}{cn(r)} \frac{\partial}{\partial r} E_\theta + \mathbf{e}_r \mathbf{e}_0 \mathbf{e}_\varphi \frac{1}{cn(r)} \frac{\partial}{\partial r} E_\varphi \\
&\quad + \mathbf{e}_r \frac{1}{n(r)} \frac{\partial}{\partial r} (\mathbf{e}_r \mathbf{e}_\theta B_\varphi) + \mathbf{e}_r \frac{1}{n(r)} \frac{\partial}{\partial r} (\mathbf{e}_\varphi \mathbf{e}_r B_\theta) + \mathbf{e}_r \frac{1}{n(r)} \frac{\partial}{\partial r} (\mathbf{e}_\theta \mathbf{e}_\varphi B_r) \\
&\quad \text{-----} \\
&\quad + \frac{\mathbf{e}_\theta}{r} \frac{\partial}{\partial \theta} \left(\mathbf{e}_0 \mathbf{e}_r \frac{E_r}{c} \right) + \frac{\mathbf{e}_\theta}{r} \frac{\partial}{\partial \theta} \left(\mathbf{e}_0 \mathbf{e}_\theta \frac{E_\theta}{c} \right) + \frac{\mathbf{e}_\theta}{r} \frac{\partial}{\partial \theta} \left(\mathbf{e}_0 \mathbf{e}_\varphi \frac{E_\varphi}{c} \right) \\
&\quad + \frac{\mathbf{e}_\theta}{r} \frac{\partial}{\partial \theta} (\mathbf{e}_r \mathbf{e}_\theta B_\varphi) + \frac{\mathbf{e}_\theta}{r} \frac{\partial}{\partial \theta} (\mathbf{e}_\varphi \mathbf{e}_r B_\theta) + \frac{\mathbf{e}_\theta}{r} \frac{\partial}{\partial \theta} (\mathbf{e}_\theta \mathbf{e}_\varphi B_r) \\
&\quad \text{-----} \\
&\quad + \mathbf{e}_\varphi \frac{1}{r \sin \theta} \frac{\partial}{\partial \varphi} \left(\mathbf{e}_0 \mathbf{e}_r \frac{E_r}{c} \right) + \mathbf{e}_\varphi \frac{1}{r \sin \theta} \frac{\partial}{\partial \varphi} \left(\mathbf{e}_0 \mathbf{e}_\theta \frac{E_\theta}{c} \right) + \mathbf{e}_\varphi \frac{1}{r \sin \theta} \frac{\partial}{\partial \varphi} \left(\mathbf{e}_0 \mathbf{e}_\varphi \frac{E_\varphi}{c} \right) \\
&\quad + \mathbf{e}_\varphi \frac{1}{r \sin \theta} \frac{\partial}{\partial \varphi} (\mathbf{e}_r \mathbf{e}_\theta B_\varphi) + \mathbf{e}_\varphi \frac{1}{r \sin \theta} \frac{\partial}{\partial \varphi} (\mathbf{e}_\varphi \mathbf{e}_r B_\theta) + \mathbf{e}_\varphi \frac{1}{r \sin \theta} \frac{\partial}{\partial \varphi} (\mathbf{e}_\theta \mathbf{e}_\varphi B_r) \quad (5.55)
\end{aligned}$$

Detailed example of the evaluation of the following term appears in Eq. (5.55)

$$\begin{aligned}
&\mathbf{e}_r \frac{1}{n} \frac{\partial}{\partial r} \left(\mathbf{e}_0 \mathbf{e}_r \frac{E_r}{c} + \mathbf{e}_0 \mathbf{e}_\theta \frac{E_\theta}{c} + \mathbf{e}_0 \mathbf{e}_\varphi \frac{E_\varphi}{c} \right. \\
&\quad \left. + \mathbf{e}_r \mathbf{e}_\theta B_\varphi + \mathbf{e}_\varphi \mathbf{e}_r B_\theta + \mathbf{e}_\theta \mathbf{e}_\varphi B_r \right) \\
&= \mathbf{e}_r \frac{1}{n} \left(\mathbf{e}_0 \mathbf{e}_r \frac{\partial}{\partial r} \frac{E_r}{c} + \frac{E_r}{c} \frac{\partial}{\partial r} \mathbf{e}_0 \mathbf{e}_r + \mathbf{e}_0 \mathbf{e}_\theta \frac{\partial}{\partial r} \frac{E_\theta}{c} + \frac{E_\theta}{c} \frac{\partial}{\partial r} \mathbf{e}_0 \mathbf{e}_\theta \right. \\
&\quad \left. + \mathbf{e}_0 \mathbf{e}_\varphi \frac{\partial}{\partial r} \frac{E_\varphi}{c} + \frac{E_\varphi}{c} \frac{\partial}{\partial r} \mathbf{e}_0 \mathbf{e}_\varphi \right) \\
&= \mathbf{e}_r \frac{1}{n} \left(\mathbf{e}_0 \mathbf{e}_r \frac{\partial}{\partial r} \frac{E_r}{c} + \frac{E_r}{c} (0) + \mathbf{e}_0 \mathbf{e}_\theta \frac{\partial}{\partial r} \frac{E_\theta}{c} + \frac{E_\theta}{c} (0) + \mathbf{e}_0 \mathbf{e}_\varphi \frac{\partial}{\partial r} \frac{E_\varphi}{c} + \frac{E_\varphi}{c} (0) \right) \\
&= \mathbf{e}_r \frac{1}{n} \left(\mathbf{e}_0 \mathbf{e}_r \frac{\partial}{\partial r} \frac{E_r}{c} + \mathbf{e}_0 \mathbf{e}_\theta \frac{\partial}{\partial r} \frac{E_\theta}{c} + \mathbf{e}_0 \mathbf{e}_\varphi \frac{\partial}{\partial r} \frac{E_\varphi}{c} \right) \\
&= -\mathbf{e}_0 \frac{1}{n} \frac{\partial}{\partial r} \frac{E_r}{c} - \mathbf{e}_0 \mathbf{e}_r \mathbf{e}_\theta \frac{1}{n} \frac{\partial}{\partial r} \frac{E_\theta}{c} + \mathbf{e}_0 \mathbf{e}_r \mathbf{e}_\varphi \frac{1}{n} \frac{\partial}{\partial r} \frac{E_\varphi}{c}
\end{aligned}$$

Evaluation of Eq. (5.55)

$$\begin{aligned}
\Box \mathbf{F} &= \mathbf{e}_r n(r) \frac{\partial E_r}{c^2 \partial t} + \mathbf{e}_\theta n(r) \frac{\partial E_\theta}{c^2 \partial t} + \mathbf{e}_\varphi n(r) \frac{\partial E_\varphi}{c^2 \partial t} \\
&\quad - \mathbf{e}_0 \mathbf{e}_r \mathbf{e}_\theta n(r) \frac{\partial B_\varphi}{c \partial t} - \mathbf{e}_0 \mathbf{e}_\varphi \mathbf{e}_r n(r) \frac{\partial B_\theta}{c \partial t} - \mathbf{e}_0 \mathbf{e}_\theta \mathbf{e}_\varphi n(r) \frac{\partial B_r}{c \partial t} \\
&\quad \text{-----} \\
&\quad - \mathbf{e}_0 \frac{1}{n(r)} \frac{\partial E_r}{c \partial r} - \mathbf{e}_0 \mathbf{e}_r \mathbf{e}_\theta \frac{1}{n(r)} \frac{\partial E_\theta}{c \partial r} + \mathbf{e}_0 \mathbf{e}_\varphi \mathbf{e}_r \frac{1}{n(r)} \frac{\partial E_\varphi}{c \partial r} \\
&\quad + \mathbf{e}_r \frac{1}{n(r)} \mathbf{e}_r \mathbf{e}_\theta \frac{\partial B_r}{\partial r} + \mathbf{e}_r \frac{1}{n(r)} \mathbf{e}_\varphi \mathbf{e}_r \frac{\partial B_\theta}{\partial r} + \mathbf{e}_r \frac{1}{n(r)} \mathbf{e}_\theta \mathbf{e}_\varphi \frac{\partial B_r}{\partial r} \\
&\quad \text{-----} \\
&\quad + \frac{\mathbf{e}_\theta \mathbf{e}_0}{r} \left(\mathbf{e}_\theta \frac{E_r}{c} + \mathbf{e}_r \frac{\partial E_r}{c \partial \theta} \right) + \frac{\mathbf{e}_\theta \mathbf{e}_0}{r} \left(-\mathbf{e}_r \frac{E_\theta}{c} + \mathbf{e}_\theta \frac{\partial E_\theta}{c \partial \theta} \right) - \frac{\mathbf{e}_0 \mathbf{e}_\theta \mathbf{e}_\varphi}{r} \left(\frac{\partial E_\varphi}{c \partial \theta} \right) \\
&\quad + \frac{\mathbf{e}_\theta}{r} \left(\mathbf{e}_\theta \mathbf{e}_\theta B_\varphi + \mathbf{e}_r (-\mathbf{e}_r) B_\varphi + \mathbf{e}_r \mathbf{e}_\theta \frac{\partial B_\varphi}{\partial \theta} \right) \\
&\quad + \frac{\mathbf{e}_\theta}{r} \left(\mathbf{e}_\varphi \mathbf{e}_\theta B_\theta - \mathbf{e}_\varphi \mathbf{e}_r B_\theta + \mathbf{e}_\varphi \mathbf{e}_r \frac{\partial B_\theta}{\partial \theta} \right) \\
&\quad + \frac{\mathbf{e}_\theta}{r} \left(-\mathbf{e}_r \mathbf{e}_\varphi B_r + \mathbf{e}_\theta \mathbf{e}_\varphi \frac{\partial B_r}{\partial \theta} \right) \\
&\quad \text{-----} \\
&\quad + \mathbf{e}_\varphi \frac{1}{r \sin \theta} \left(\mathbf{e}_0 \mathbf{e}_\varphi \sin \theta \frac{E_r}{c} + \mathbf{e}_0 \mathbf{e}_r \frac{\partial E_r}{c \partial \varphi} \right) \\
&\quad + \mathbf{e}_\varphi \frac{1}{r \sin \theta} \left(\mathbf{e}_0 \mathbf{e}_\varphi \cos \theta \frac{E_\theta}{c} + \mathbf{e}_0 \mathbf{e}_\theta \frac{\partial E_\theta}{c \partial \varphi} \right) \\
&\quad + \mathbf{e}_\varphi \frac{1}{r \sin \theta} \left(-\mathbf{e}_0 \mathbf{e}_\theta \cos \theta - \mathbf{e}_0 \mathbf{e}_r \sin \theta \frac{E_\varphi}{c} + \mathbf{e}_\varphi \mathbf{e}_0 \mathbf{e}_\varphi \frac{1}{r \sin \theta} \frac{\partial E_\varphi}{c \partial \varphi} \right) \\
&\quad + \mathbf{e}_\varphi \frac{1}{r \sin \theta} \left(\mathbf{e}_\varphi \sin \theta \mathbf{e}_\theta B_\varphi + \mathbf{e}_r \mathbf{e}_\varphi \cos \theta B_\varphi + \mathbf{e}_r \mathbf{e}_\theta \frac{\partial B_\varphi}{\partial \varphi} \right) \\
&\quad \text{-----} \\
&\quad + \mathbf{e}_\varphi \frac{1}{r \sin \theta} \left((-\mathbf{e}_\theta \cos \theta - \mathbf{e}_r \sin \theta) \mathbf{e}_r B_\theta + \mathbf{e}_\varphi \mathbf{e}_\varphi \sin \theta B_\theta + \mathbf{e}_\varphi \mathbf{e}_r \frac{\partial B_\theta}{\partial \varphi} \right) \\
&\quad + \mathbf{e}_\varphi \frac{1}{r \sin \theta} \left(\mathbf{e}_\varphi \cos \theta \mathbf{e}_\varphi B_r + \mathbf{e}_\theta (-\mathbf{e}_\theta \cos \theta - \mathbf{e}_r \sin \theta) B_r + \mathbf{e}_\theta \mathbf{e}_\varphi \frac{\partial B_r}{\partial \varphi} \right)
\end{aligned}$$

Rewriting

$$\begin{aligned}
\Box \mathbf{F} &= \mathbf{e}_r n(r) \frac{\partial E_r}{c^2 \partial t} + \mathbf{e}_\theta n(r) \frac{\partial E_\theta}{c^2 \partial t} + \mathbf{e}_\varphi n(r) \frac{\partial E_\varphi}{c^2 \partial t} \\
&\quad - \mathbf{e}_0 \mathbf{e}_r \mathbf{e}_\theta n(r) \frac{\partial B_\varphi}{c \partial t} + \mathbf{e}_0 \mathbf{e}_r \mathbf{e}_\varphi n(r) \frac{\partial B_\theta}{c \partial t} - \mathbf{e}_0 \mathbf{e}_\theta \mathbf{e}_\varphi n(r) \frac{\partial B_r}{c \partial t} \\
&\quad \text{-----} \\
&\quad - \mathbf{e}_0 \frac{1}{n(r)} \frac{\partial E_r}{c \partial r} - \mathbf{e}_0 \mathbf{e}_r \mathbf{e}_\theta \frac{1}{n(r)} \frac{\partial E_\theta}{c \partial r} + \mathbf{e}_0 \mathbf{e}_r \mathbf{e}_\varphi \frac{1}{n(r)} \frac{\partial E_\varphi}{c \partial r} \\
&\quad + \mathbf{e}_\theta \frac{1}{n(r)} \frac{\partial B_r}{\partial r} - \mathbf{e}_\varphi \frac{1}{n(r)} \frac{\partial B_\theta}{\partial r} + \mathbf{e}_r \mathbf{e}_\theta \mathbf{e}_\varphi \frac{1}{n(r)} \frac{\partial B_r}{\partial r} \\
&\quad \text{-----} \\
&\quad - \mathbf{e}_0 \frac{1}{r} \frac{E_r}{c} + \mathbf{e}_0 \mathbf{e}_r \mathbf{e}_\theta \frac{1}{c} \frac{\partial E_r}{r \partial \theta} - \mathbf{e}_0 \mathbf{e}_r \mathbf{e}_\theta \frac{1}{r} \frac{E_\theta}{c} - \mathbf{e}_0 \frac{1}{r} \frac{\partial E_\theta}{c \partial \theta} - \mathbf{e}_0 \mathbf{e}_\theta \mathbf{e}_\varphi \frac{1}{r} \frac{\partial E_\varphi}{c \partial \theta} \\
&\quad + \mathbf{e}_\theta \frac{1}{r} B_\varphi - \mathbf{e}_\theta \frac{B_\varphi}{r} - \mathbf{e}_r \frac{1}{r} \frac{\partial B_\varphi}{\partial \theta} - \mathbf{e}_\varphi \frac{1}{r} B_\theta - \mathbf{e}_r \mathbf{e}_\theta \mathbf{e}_\varphi \frac{B_\theta}{r} + \mathbf{e}_r \mathbf{e}_\theta \mathbf{e}_\varphi \frac{1}{r} \frac{\partial B_\theta}{\partial \theta} \\
&\quad + \mathbf{e}_r \mathbf{e}_\theta \mathbf{e}_\varphi \frac{1}{r} B_r + \mathbf{e}_\varphi \frac{1}{r} \frac{\partial B_r}{\partial \theta} \\
&\quad \text{-----} \\
&\quad - \mathbf{e}_0 \frac{\sin \theta}{r \sin \theta} \frac{E_r}{c} + \mathbf{e}_0 \mathbf{e}_r \mathbf{e}_\varphi \frac{1}{r \sin \theta} \frac{\partial E_r}{c \partial \varphi} \\
&\quad - \mathbf{e}_0 \frac{1}{r \sin \theta} \cos \theta \frac{E_\theta}{c} + \mathbf{e}_0 \mathbf{e}_\theta \mathbf{e}_\varphi \frac{1}{r \sin \theta} \frac{1}{c} \frac{\partial E_\theta}{\partial \varphi} \\
&\quad + \frac{\cos \theta}{r \sin \theta} (-\mathbf{e}_0 \mathbf{e}_\theta \mathbf{e}_\varphi) E_\varphi - \mathbf{e}_0 \mathbf{e}_r \mathbf{e}_\varphi \frac{\sin \theta}{r \sin \theta} \frac{E_\varphi}{c} - \mathbf{e}_\varphi \frac{1}{r \sin \theta} \frac{\partial E_\varphi}{\partial \varphi} \\
&\quad - \mathbf{e}_\theta \frac{\sin \theta}{r \sin \theta} B_\varphi - \frac{\mathbf{e}_r \cos \theta B_\varphi}{r \sin \theta} + \mathbf{e}_r \mathbf{e}_\theta \mathbf{e}_\varphi \frac{1}{r \sin \theta} \frac{\partial B_\varphi}{\partial \varphi} \\
&\quad + \mathbf{e}_r \mathbf{e}_\theta \mathbf{e}_\varphi \frac{\cos \theta B_\theta}{r \sin \theta} - \mathbf{e}_\varphi \frac{\sin \theta}{r \sin \theta} B_\theta + \mathbf{e}_\varphi \frac{\sin \theta}{r \sin \theta} B_\theta + \mathbf{e}_r \frac{1}{r \sin \theta} \frac{\partial B_\theta}{\partial \varphi} \\
&\quad + \mathbf{e}_\varphi \frac{\cos \theta B_r}{r \sin \theta} - \mathbf{e}_\varphi \frac{\cos \theta B_r}{r \sin \theta} + \mathbf{e}_r \mathbf{e}_\theta \mathbf{e}_\varphi \frac{\sin \theta B_r}{r \sin \theta} - \mathbf{e}_\theta \frac{1}{r \sin \theta} \frac{\partial B_r}{\partial \varphi} \\
&\quad - \mathbf{e}_0 \frac{1}{n(r)} \frac{\partial E_r}{c \partial r} - \mathbf{e}_0 \frac{\partial E_\theta}{c r \partial \theta} - \mathbf{e}_0 \frac{1}{c} \frac{\partial E_\varphi}{r \sin \theta \partial \varphi} \\
&\quad + \mathbf{e}_r \left(\frac{n(r)}{c^2} \frac{\partial E_r}{\partial t} + \frac{1}{r \sin \theta} \frac{\partial B_\theta}{\partial \varphi} - \frac{1}{r} \frac{\partial B_\varphi}{\partial \theta} \right) \\
&\quad + \mathbf{e}_\theta \left(n(r) \frac{\partial E_\theta}{c^2 \partial t} + \frac{1}{n(r)} \frac{\partial B_r}{\partial r} - \frac{1}{r \sin \theta} \frac{\partial B_r}{\partial \varphi} \right)
\end{aligned}$$

$$\begin{aligned}
& +\mathbf{e}_\varphi \left(n(r) \frac{\partial E_\varphi}{c^2 \partial t} + \frac{\partial B_r}{r \partial \theta} - \frac{\partial B_\theta}{n(r) \partial r} \right) \\
& +\mathbf{e}_0 \mathbf{e}_r \mathbf{e}_\theta \frac{1}{c} \left(\frac{\partial E_r}{r \partial \theta} - \frac{1}{c n(r)} \frac{\partial E_\theta}{\partial r} - \frac{n(r) \partial B_\varphi}{c \partial t} \right) \\
& +\mathbf{e}_0 \mathbf{e}_r \mathbf{e}_\varphi \frac{1}{c} \left(\frac{1}{n(r)} \frac{\partial E_\varphi}{\partial r} - \frac{\partial E_r}{r \sin \theta \partial \varphi} - \frac{n(r) \partial B_\theta}{\partial t} \right) \\
& +\mathbf{e}_0 \mathbf{e}_\theta \mathbf{e}_\varphi \frac{1}{c} \left(\frac{\partial E_\theta}{r \sin \theta \partial \varphi} - \frac{\partial E_\varphi}{r \partial \theta} - \frac{n(r) \partial B_r}{\partial t} \right) \\
& +\mathbf{e}_r \mathbf{e}_\theta \mathbf{e}_\varphi \left(\frac{\partial B_r}{n(r) \partial r} + \frac{\partial B_\theta}{r \partial \theta} + \frac{1}{r \sin \theta} \frac{\partial B_\varphi}{\partial \varphi} \right) \\
& -\mathbf{e}_0 \left(\frac{1}{r} \frac{E_r}{c} + \frac{1}{r} \frac{E_r}{c} \right) - \mathbf{e}_0 \frac{\cos \theta}{r \sin \theta} \frac{E_\theta}{c} - \mathbf{e}_\theta \left(\frac{1}{r} B_\varphi + \frac{B_\varphi}{r} \right) \\
& -\mathbf{e}_\varphi B_\theta \left(\frac{1}{r} + \frac{\sin \theta}{r \sin \theta} - \frac{\sin \theta}{r \sin \theta} \right) + \mathbf{e}_\varphi \frac{B_r}{r} \left(\frac{\cos \theta}{\sin \theta} - \frac{\cos \theta}{\sin \theta} \right) \\
& -\mathbf{e}_r \frac{\cos \theta B_\varphi}{r \sin \theta} - \mathbf{e}_0 \mathbf{e}_r \mathbf{e}_\theta \frac{1}{r} \frac{E_\theta}{c} - \mathbf{e}_0 \mathbf{e}_\theta \mathbf{e}_\varphi \frac{\cos \theta}{r \sin \theta} \frac{E_\varphi}{c} - \mathbf{e}_0 \mathbf{e}_r \mathbf{e}_\varphi \frac{1}{r} \frac{E_\varphi}{c} \\
& +\mathbf{e}_r \mathbf{e}_\theta \mathbf{e}_\varphi \left(\frac{B_r}{r} + \frac{\sin \theta B_r}{r \sin \theta} - \frac{B_\theta}{r} + \frac{\cos \theta B_\theta}{r \sin \theta} \right) \\
& -\mathbf{e}_0 \left(\frac{2 \partial E_r}{rc} + \frac{\cos \theta}{r \sin \theta} \frac{E_\theta}{c} + \frac{1}{n(r)} \frac{\partial E_r}{c \partial r} + \frac{1}{r} \frac{\partial E_\theta}{c \partial \theta} \right) \\
& +\mathbf{e}_r \left(n(r) \frac{\partial E_r}{c^2 \partial t} - \frac{1}{r} \frac{\partial B_\varphi}{\partial \theta} - \frac{\cos \theta B_\varphi}{r \sin \theta} \right) \\
& +\mathbf{e}_\theta \left(n(r) \frac{\partial E_\theta}{c^2 \partial t} + \frac{1}{n(r)} \frac{\partial B_r}{\partial r} + \frac{B_\varphi}{r} - \frac{B_\varphi}{r} - \frac{\sin \theta B_\theta}{r \cos \theta} - \frac{1}{r \sin \theta} \frac{\partial B_r}{\partial \varphi} \right) \\
& +\mathbf{e}_\varphi \left(n(r) \frac{\partial E_\varphi}{c \partial t} - \frac{B_\theta}{r} + \frac{1}{r} \frac{\partial B_r}{\partial \theta} - \frac{\sin \theta B_\theta}{r \sin \theta} + \frac{\sin \theta B_\theta}{r \sin \theta} + \frac{\cos \theta B_r}{r \sin \theta} - \frac{\cos \theta B_r}{r \sin \theta} \right) \\
& +\mathbf{e}_0 \mathbf{e}_r \mathbf{e}_\theta \left(-n(r) \frac{\partial B_\varphi}{c \partial t} - \frac{1}{n(r)} \frac{\partial E_\theta}{c \partial r} + \frac{1}{cr} \frac{\partial E_r}{\partial \theta} - \frac{1}{r} \frac{E_\theta}{c} \right) \\
& +\mathbf{e}_0 \mathbf{e}_r \mathbf{e}_\varphi \left(n(r) \frac{\partial B_\theta}{c \partial t} + \frac{1}{n(r)} \frac{\partial E_\varphi}{c \partial r} - \frac{\sin \theta}{r \sin \theta} \frac{E_\varphi}{c} \right) \\
& +\mathbf{e}_0 \mathbf{e}_\theta \mathbf{e}_\varphi \left(-n(r) \frac{\partial B_r}{c \partial t} - \frac{1}{r} \frac{\partial E_\varphi}{c \partial \theta} + \frac{1}{r \sin \theta} \frac{1}{c} \frac{\partial E_\theta}{\partial \varphi} \right) \\
& +\mathbf{e}_r \mathbf{e}_\theta \mathbf{e}_\varphi \left(\frac{1}{n(r)} \frac{\partial B_r}{\partial r} + \frac{\partial B_\theta}{r \partial \theta} + \frac{1}{r \sin \theta} \frac{\partial B_\varphi}{\partial \varphi} + \frac{B_r}{r} + \frac{\sin \theta B_r}{r \sin \theta} - \frac{B_\theta}{r} + \frac{\cos \theta B_\theta}{r \sin \theta} \right)
\end{aligned}$$

The terms of \mathbf{A} components

$$\begin{aligned}\frac{E_r}{c} &= -n(r) \frac{\partial A_r}{c \partial t} - \frac{1}{n(r)} \frac{\partial A_0}{\partial r} \\ \frac{E_\theta}{c} &= -n(r) \frac{\partial A_\theta}{c \partial t} - \frac{1}{r} \frac{\partial A_\theta}{\partial r} \\ \frac{E_\varphi}{c} &= -n(r) \frac{\partial A_\varphi}{c \partial t} - \frac{1}{r} \frac{\partial A_\theta}{\partial r} \\ B_{r\theta} &= \frac{1}{n(r)} \left(\frac{\partial A_\theta}{\partial r} + A_r \right) - \frac{1}{r} \left(\frac{\partial A_r}{\partial \theta} - A_\theta \right) \\ B_{\varphi r} &= \frac{1}{r \sin \theta} \left(\frac{\partial A_r}{\partial \varphi} - A_\varphi \sin \theta - \frac{\partial A_\varphi}{\partial r} \right) \\ B_{\theta\varphi} &= \frac{1}{r} \frac{\partial A_\varphi}{\partial r} - \frac{1}{r \sin \theta} \left(\frac{\partial A_\theta}{\partial \varphi} - A_\varphi \cos \theta \right)\end{aligned}$$

Add source terms to the right hand side of the above. The preceding sketches the basic structure of Maxwell's Equations in a Schwarzschild metric. Results are applicable to spherical coordinates in flat space by putting $n(r) = 1$.

5.5 Addendum: Laplacian Gradient in Schwarzschild Metric

$$\begin{aligned}d\mathbf{s} &= \mathbf{e}_r h_r dr + \mathbf{e}_\theta h_\theta d\theta + \mathbf{e}_\varphi h_\varphi d\varphi \\ d\mathbf{s} &= \mathbf{e}_r \frac{dr}{(1 - 2\alpha/r)^{1/2}} + \mathbf{e}_\theta r d\theta + \mathbf{e}_\varphi r \sin \theta d\varphi \\ h_r &= \frac{1}{(1 - 2\alpha/r)^{1/2}} \quad h_\theta = r \quad h_\varphi = r \sin \theta \\ \text{grad} &= \mathbf{e}_r \frac{1}{h_r} \frac{\partial}{\partial r} + \mathbf{e}_\theta \frac{1}{h_\theta} \frac{\partial}{\partial \theta} + \mathbf{e}_\varphi \frac{1}{h_\varphi} \frac{\partial}{\partial \varphi}\end{aligned}$$

Thus for the spatial part of Schwarzschild metric, the gradient is

$$\begin{aligned}\text{grad}\psi &= \left(\mathbf{e}_r \frac{1}{h_r} \frac{\partial}{\partial r} + \mathbf{e}_\theta \frac{1}{h_\theta} \frac{\partial}{\partial \theta} + \mathbf{e}_\varphi \frac{1}{h_\varphi} \frac{\partial}{\partial \varphi} \right) \psi = (\mathbf{e}_r g_r + \mathbf{e}_\theta g_\theta + \mathbf{e}_\varphi g_\varphi) \psi \\ g_r &= (1 - 2\alpha/r)^{1/2} \frac{\partial}{\partial r} \quad g_\theta = \frac{1}{r} \frac{\partial}{\partial \theta} \quad g_\varphi = \frac{1}{r \sin \theta} \frac{\partial}{\partial \varphi}\end{aligned}$$

The divergence in spherical coordinates is

$$\text{div} \mathbf{g} = \frac{1}{r^2 \sin \varphi} \left[\sin \theta \frac{\partial}{\partial r} (r^2 g_r) + r \frac{\partial}{\partial \theta} (\sin \theta g_\theta) + r \frac{\partial g_\varphi}{\partial \varphi} \right]$$

The Laplacian in spherical coordinates is ($2\alpha/r = 0$)

$$\nabla^2\psi = \operatorname{div} \operatorname{grad}\psi = \frac{1}{r^2 \sin \theta} \left[\sin \theta \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} \right) + \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin \theta} \frac{\partial^2}{\partial \varphi^2} \right] \psi$$

The Laplacian in Schwarzschild metric is

$$\nabla^2\psi = \frac{1}{r^2 \sin \theta} \left[\sin \theta \frac{\partial}{\partial r} \left[r^2 \left(1 - \frac{2\alpha}{r} \right) \frac{\partial}{\partial r} \right] + \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \varphi^2} \right] \psi$$

$$\frac{\partial}{\partial r} \left[r^2 \left(1 - \frac{2\alpha}{r} \right) \frac{\partial}{\partial r} \right] = 2r \left(1 - \frac{2\alpha}{r} \right) \frac{\partial}{\partial r} + r^2 \left(\frac{1}{r} \frac{\alpha}{r} \right) \frac{\partial}{\partial r} + r^2 \left(1 - \frac{\alpha}{r} \right) \frac{\partial}{\partial r^2}$$

$$\frac{2\alpha}{r} = \text{dimensionless} \quad \frac{1}{(1 - 2\alpha/r)^{1/2}} \cong -\frac{1}{1 - \alpha/r} = 1 + \alpha/r$$

5.6 Standard 3-D Spherical Coordinates

$$ds = \mathbf{e}_r h_r dr + \mathbf{e}_\theta h_\theta d\theta + \mathbf{e}_\varphi h_\varphi d\varphi$$

$$ds = \frac{\mathbf{e}_r dr}{(1 - 2\alpha/r)} + \mathbf{e}_\theta r d\theta + \mathbf{e}_\varphi r \sin \theta d\varphi$$

In the Schwarzschild metric

$$\operatorname{grad}\psi = \left[\mathbf{e}_r \frac{1}{h_r} \frac{\partial}{\partial r} + \mathbf{e}_\theta \frac{1}{h_\theta} \frac{\partial}{\partial \theta} + \mathbf{e}_\varphi \frac{1}{h_\varphi} \frac{\partial}{\partial \varphi} \right] \psi$$

$$\operatorname{grad}\psi = \left[\mathbf{e}_r \frac{1}{(1 - 2\alpha/r)} \frac{\partial}{\partial r} + \mathbf{e}_\theta \frac{\partial}{r \partial \theta} + \mathbf{e}_\varphi \frac{1}{r \sin \theta} \frac{\partial}{\partial \varphi} \right] \psi$$

$$= (\mathbf{e}_r g_r + \mathbf{e}_\theta g_\theta + \mathbf{e}_\varphi g_\varphi) \psi$$

$$g_r = \frac{1}{(1 - 2\alpha/r)} \frac{\partial}{\partial r} \quad g_\theta = \frac{1}{r} \frac{\partial}{\partial \theta} \quad g_\varphi = \frac{1}{r \sin \theta} \frac{\partial}{\partial \varphi}$$

$$\operatorname{div}\Theta = \frac{1}{r^2 \sin \theta} \left[\sin \theta \frac{\partial}{\partial r} (r^2 V_r) + r \frac{\partial}{\partial \theta} (\sin \theta V_\theta) + r^2 \frac{\partial V_\varphi}{\partial \varphi} \right]$$

5.7 Evaluation of $\widehat{\mathbf{F}}\square\mathbf{F} + \mathbf{F}\square\widehat{\mathbf{F}}$

In this exercise we evaluate $\widehat{\mathbf{F}}\square\mathbf{F} + \mathbf{F}\square\widehat{\mathbf{F}}$ in the form given by Eq. (5.56).

$$\begin{aligned}\widehat{\mathbf{F}}\square\mathbf{F} + \mathbf{F}\square\widehat{\mathbf{F}} &= \left(\mathbf{F}\mathbf{e}_\mu^{-1} \frac{\partial}{\partial x_\mu} \right) \mathbf{F} + \mathbf{F} \left(\mathbf{e}_\mu^{-1} \frac{\partial \mathbf{F}}{\partial x_\mu} \right) \quad \mu = 0, 1, 2, 3. \quad (5.56) \\ \square &= -\mathbf{e}_0 \frac{\partial}{c\partial t} + \mathbf{e}_1 \frac{\partial}{\partial x_1} + \mathbf{e}_2 \frac{\partial}{\partial x_2} + \mathbf{e}_3 \frac{\partial}{\partial x_3} \\ &= \mathbf{e}_0^{-1} \frac{\partial}{\partial x_0} + \mathbf{e}_1^{-1} \frac{\partial}{\partial x_1} + \mathbf{e}_2^{-1} \frac{\partial}{\partial x_2} + \mathbf{e}_3^{-1} \frac{\partial}{\partial x_3}\end{aligned}$$

Since $\mathbf{e}_0^{-1} = -\mathbf{e}_0$ and $\mathbf{e}_i^{-1} = \mathbf{e}_i$ $i = 1, 2, 3$, $x_0 = ct$.

\mathbf{F} is the space-time bivector shown in Eq. (5.58).

Note that Eq. (5.56) may also be written

$$\widehat{\mathbf{F}}\square\mathbf{F} + \mathbf{F}\square\widehat{\mathbf{F}} = \frac{\partial}{\partial x_\mu} (\mathbf{F}\mathbf{e}_\mu^{-1}\mathbf{F}) \quad (5.57)$$

One may evaluate $\mathbf{F}\mathbf{e}_\mu^{-1}\mathbf{F}$ in Eq. (5.57) and then perform the differentiation as indicated. In this section we evaluate the left hand side of Eq. (5.57) which turns out to be somewhat simpler than evaluation of the right-hand side of Eq. (5.57).

$$\begin{aligned}\mathbf{F}\square &= \left(\mathbf{e}_0\mathbf{e}_1 \frac{E_x}{c} + \mathbf{e}_0\mathbf{e}_2 \frac{E_y}{c} + \mathbf{e}_0\mathbf{e}_3 \frac{E_z}{c} + \mathbf{e}_1\mathbf{e}_2 B_z + \mathbf{e}_3\mathbf{e}_1 B_y + \mathbf{e}_2\mathbf{e}_3 B_x \right) \\ &\quad \times \left(-\mathbf{e}_0 \frac{\partial}{c\partial t} + \mathbf{e}_1 \frac{\partial}{\partial x} + \mathbf{e}_2 \frac{\partial}{\partial y} + \mathbf{e}_3 \frac{\partial}{\partial z} \right) \quad (5.58) \\ \mathbf{F}\square &= \mathbf{e}_0 \operatorname{div} \frac{\mathbf{E}}{c} + \mathbf{e}_1 \left[-\frac{\partial E_x}{c^2 \partial t} + (\operatorname{curl} \mathbf{B})_x \right] + \mathbf{e}_2 \left[-\frac{\partial E_y}{c^2 \partial t} + (\operatorname{curl} \mathbf{B})_y \right] \\ &\quad + \mathbf{e}_3 \left[-\frac{\partial E_z}{c^2 \partial t} + (\operatorname{curl} \mathbf{B})_z \right] + \mathbf{e}_0\mathbf{e}_1\mathbf{e}_2 \left[-\frac{\partial B_z}{c\partial t} - (\operatorname{curl} \mathbf{E})_z \right] \\ &\quad + \mathbf{e}_0\mathbf{e}_3\mathbf{e}_1 \left[-\frac{\partial B_y}{c\partial t} - (\operatorname{curl} \mathbf{E})_y \right] + \mathbf{e}_0\mathbf{e}_2\mathbf{e}_3 \left[-\frac{\partial B_x}{c\partial t} - (\operatorname{curl} \mathbf{E})_x \right] \\ &\quad + \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 \operatorname{div} \mathbf{B} \quad (5.59)\end{aligned}$$

$$\begin{aligned} \mathbf{F}\square &= \mathbf{e}_0 \operatorname{div} \frac{\mathbf{E}}{c} - \frac{\partial \mathbf{E}}{c^2 \partial t} + \operatorname{curl} \mathbf{B} + \mathbf{e}_5 \left[\mathbf{e}_3 \left(-\frac{\partial B_z}{c \partial t} - (\operatorname{curl} \mathbf{E})_z \right) \right. \\ &\quad \left. + \mathbf{e}_2 \left(-\frac{\partial B_y}{c \partial t} - (\operatorname{curl} \mathbf{E})_y \right) + \mathbf{e}_1 \left(-\frac{\partial B_x}{c \partial t} - (\operatorname{curl} \mathbf{E})_x \right) + \mathbf{e}_0 (\operatorname{div} \mathbf{B}) \right] \end{aligned} \quad (5.60)$$

$$\begin{aligned} \mathbf{F}\square &= \mathbf{e}_0 \operatorname{div} \frac{\mathbf{E}}{c} - \frac{\partial \mathbf{E}}{c^2 \partial t} + \operatorname{curl} \mathbf{B} + \mathbf{e}_5 \left[\mathbf{e}_0 \operatorname{div} \mathbf{B} - \frac{\partial \mathbf{B}}{c \partial t} - \operatorname{curl} \mathbf{E} \right] \\ &= \mathbf{V} + \mathbf{T} = \mathbf{V} + \mathbf{e}_5 \mathbf{U} \end{aligned} \quad (5.61)$$

Note that the trivector part of Eq. (5.61) is the same in sign to the trivector part of Eq. (5.68). The vector parts are of a trivector may be written as follows, where $\mathbf{e}_5 = \mathbf{e}_0 \mathbf{e}_1 \mathbf{e}_2 \mathbf{e}_3$ and $\mathbf{e}_5 \mathbf{e}_5 = 1$.

$$\begin{aligned} \mathbf{T} &= T^{012} \mathbf{e}_0 \mathbf{e}_1 \mathbf{e}_2 + T^{031} \mathbf{e}_0 \mathbf{e}_3 \mathbf{e}_1 + T^{023} \mathbf{e}_0 \mathbf{e}_2 \mathbf{e}_3 + T^{123} \mathbf{e}_1 \mathbf{e}_2 \mathbf{e}_3 \\ -\mathbf{e}_5 \mathbf{e}_5 \mathbf{T} &= \mathbf{T} = \mathbf{e}_5 (-\mathbf{e}_5 \mathbf{T}) = \mathbf{e}_5 (T^{012} \mathbf{e}_3 + T^{031} \mathbf{e}_2 + T^{023} \mathbf{e}_1 + T^{123} \mathbf{e}_0) \\ \mathbf{T} &= \mathbf{e}_5 \mathbf{U} \\ \mathbf{U} &= T^{123} \mathbf{e}_0 + T^{023} \mathbf{e}_1 + T^{031} \mathbf{e}_2 + T^{012} \mathbf{e}_3 \\ &= u_0 \mathbf{e}_0 + u_1 \mathbf{e}_1 + u_2 \mathbf{e}_2 + u_3 \mathbf{e}_3 \\ u_0 &= T^{123}, \quad u_1 = T^{023}, \quad u_2 = T^{031}, \quad u_3 = T^{012} \end{aligned}$$

When $\mathbf{F}\square$ in Eq. (5.59), a vector, is put equal to $\mu_0 \mathbf{J}_s$, where \mathbf{J}_s is the 4-current density

$$\mathbf{J}_s = (c\rho_s \mathbf{e}_0 + j_{sx} \mathbf{e}_1 + j_{sy} \mathbf{e}_2 + j_{sz} \mathbf{e}_3) = (c\rho_s \mathbf{e}_0 + \mathbf{j}_s) \quad (5.62)$$

or if the field is produced by a single charge q_s moving with velocity \mathbf{v}_s

$$\mathbf{V}_s = q_s (c\mathbf{e}_0 + v_{sx} \mathbf{e}_1 + v_{sy} \mathbf{e}_2 + v_{sz} \mathbf{e}_3) = q_s (c\mathbf{e}_0 + \mathbf{v}_s) = q_s \mathbf{V}_s \quad (5.63)$$

one obtains Maxwell's equations. The trivector part is normally put equal to zero, rather than equated to the trivector $\mathbf{e}_5 \mathbf{U}_s$. $\mathbf{F}\square$ is normally equated to zero since the trivector part of $\mathbf{F}\square$ is identically zero when \mathbf{F} is expressed in terms of a vector potential via $\mathbf{F} = \square \mathbf{A}$, where \mathbf{A} is a space-time vector. Now equate Eqs. (5.61) and (5.62) and then equate the vector parts to obtain:

$$\left[\mathbf{e}_0 \operatorname{div} \frac{\mathbf{E}_s}{c} - \frac{1}{c^2} \frac{\partial \mathbf{E}_s}{\partial t} + \operatorname{curl} \mathbf{B}_s \right] = \mu_0 (c\rho_s \mathbf{e}_0 + \mathbf{j}_s) \quad (5.64)$$

We drop the γ_s factor on the right hand side of Eq. (5.64) so that the equations appear in their more usual form. The factor γ_s can be inserted at any time.

Equating the trivector parts:

$$\mathbf{e}_5 \left(\mathbf{e}_0 \operatorname{div} \mathbf{B}_s - \frac{\partial \mathbf{B}_s}{\partial t} - \operatorname{curl} \frac{\mathbf{E}_s}{c} \right) = \mathbf{e}_5 \mathbf{U} \quad (5.65)$$

Equate the time and spatial parts of Eq. (5.64):

$$\begin{aligned} \operatorname{div} \frac{\mathbf{E}_s}{c} &= \mu_0 c \rho_s = \frac{\rho_s}{\varepsilon_0} \quad \text{using } \mu_0 \varepsilon_0 = 1/c^2 \\ -\frac{1}{c^2} \frac{\partial \mathbf{E}_s}{\partial t} + \operatorname{curl} \mathbf{B}_s &= \mu_0 \mathbf{j}_s \end{aligned}$$

Do the same for the trivector terms in Eq. (5.65). Then

$$\begin{aligned} \operatorname{div} \mathbf{B}_s &= 0 \\ \frac{\partial \mathbf{B}_s}{\partial t} + \operatorname{curl} \frac{\mathbf{E}_s}{c} &= 0 \end{aligned}$$

The subscript s indicates that the field quantities arise from the source current \mathbf{J}_s or the single charge current $q_s \mathbf{V}_s$. As stated, $\mathbf{e}_5 \mathbf{U}$ is identically zero when $\mathbf{F} = \square \mathbf{A}$.

Omitting the subscripts,

$$\mathbf{F}\square = \mathbf{V} + \mathbf{T} = \mathbf{V} + \mathbf{e}_5 \mathbf{U}$$

and

$$\square\mathbf{F} = -\mathbf{V} + \mathbf{T} = -\mathbf{V} - \mathbf{e}_5 \mathbf{e}_5 \mathbf{T} = -\mathbf{V} + \mathbf{e}_5 (-\mathbf{e}_5 \mathbf{T})$$

$$\square\mathbf{F} = -\mathbf{V} + \mathbf{e}_5 \mathbf{U}$$

$$\begin{aligned} \square\mathbf{F} &= \left(-\mathbf{e}_0 \frac{\partial}{c \partial t} + \mathbf{e}_1 \frac{\partial}{\partial x} + \mathbf{e}_2 \frac{\partial}{\partial y} + \mathbf{e}_3 \frac{\partial}{\partial z} \right) \times \\ &\quad \left(\mathbf{e}_0 \mathbf{e}_1 \frac{E_x}{c} + \mathbf{e}_0 \mathbf{e}_2 \frac{E_y}{c} + \mathbf{e}_0 \mathbf{e}_3 \frac{E_z}{c} + \mathbf{e}_1 \mathbf{e}_2 B_z + \mathbf{e}_3 \mathbf{e}_1 B_y + \mathbf{e}_2 \mathbf{e}_3 B_x \right) \\ &= -\mathbf{e}_0 \left(\frac{\partial}{\partial x} \frac{E_x}{c} + \frac{\partial}{\partial y} \frac{E_y}{c} + \frac{\partial}{\partial z} \frac{E_z}{c} \right) + \mathbf{e}_1 \left[\frac{1}{c^2} \frac{\partial E_x}{\partial t} - (\operatorname{curl} B)_x \right] \\ &\quad + \mathbf{e}_2 \left[\frac{1}{c^2} \frac{\partial E_y}{\partial t} - (\operatorname{curl} B)_y \right] + \mathbf{e}_3 \left[\frac{1}{c^2} \frac{\partial E_z}{\partial t} - (\operatorname{curl} B)_z \right] \\ &\quad + \mathbf{e}_0 \mathbf{e}_1 \mathbf{e}_2 \left[-\frac{\partial B_z}{c \partial t} - \left(\operatorname{curl} \frac{E}{c} \right)_z \right] + \mathbf{e}_0 \mathbf{e}_3 \mathbf{e}_1 \left[-\frac{\partial B_y}{c \partial t} - \left(\operatorname{curl} \frac{E}{c} \right)_y \right] \\ &\quad + \mathbf{e}_0 \mathbf{e}_2 \mathbf{e}_3 \left[-\frac{\partial B_x}{c \partial t} - \left(\operatorname{curl} \frac{E}{c} \right)_x \right] + \mathbf{e}_1 \mathbf{e}_2 \mathbf{e}_3 \left[\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} \right] \quad (5.66) \end{aligned}$$

$$\begin{aligned} \square \mathbf{F} &= -\mathbf{e}_0 \operatorname{div} \frac{\mathbf{E}}{c} + \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t} - \operatorname{curl} \mathbf{B} + \mathbf{e}_5 \left[\mathbf{e}_3 \left(-\frac{\partial B_z}{c \partial t} - \left(\operatorname{curl} \frac{\mathbf{E}}{c} \right)_z \right) \right. \\ &\quad \left. + \mathbf{e}_2 \left(-\frac{\partial B_y}{c \partial t} - \left(\operatorname{curl} \frac{\mathbf{E}}{c} \right)_y \right) + \mathbf{e}_1 \left(-\frac{\partial B_x}{c \partial t} - \left(\operatorname{curl} \frac{\mathbf{E}}{c} \right)_x \right) + \mathbf{e}_0 \operatorname{div} \mathbf{B} \right] \end{aligned} \quad (5.67)$$

$$= -\mathbf{e}_0 \operatorname{div} \frac{\mathbf{E}}{c} + \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t} - \operatorname{curl} \mathbf{B} + \mathbf{e}_5 \left[\mathbf{e}_0 \operatorname{div} \mathbf{B} - \frac{\partial \mathbf{B}}{c \partial t} - \operatorname{curl} \frac{\mathbf{E}}{c} \right] \quad (5.68)$$

$$= \operatorname{vec} - \operatorname{triv} = \mathbf{V} - \mathbf{T} = \mathbf{V} - \mathbf{e}_5 \mathbf{U} \quad \mathbf{e}_5 = \mathbf{e}_0 \mathbf{e}_2 \mathbf{e}_3, \quad \mathbf{e}_5 \mathbf{e}_5 = -1$$

Note that the trivector part of Eq. (5.61) is same in sign to the trivector part of Eq. (5.68). The vector parts are opposite in sign. Equating corresponding multivector parts of Eq. (5.68) to the source currents

$$\begin{aligned} \mathbf{V} &= -\mathbf{e}_0 \operatorname{div} \frac{\mathbf{E}}{c} + \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t} - \operatorname{curl} \mathbf{B} \\ \mathbf{T} &= \mathbf{e}_5 \mathbf{U} = \mathbf{e}_5 \left[\mathbf{e}_0 \operatorname{div} \mathbf{B} - \frac{\partial \mathbf{B}}{\partial t} - \operatorname{curl} \frac{\mathbf{E}}{c} \right] \end{aligned}$$

Thus

$$\widehat{\mathbf{F}} \square = \mathbf{V} + \mathbf{T} = \mathbf{V} + \mathbf{e}_5 \mathbf{U} \quad (5.69)$$

$$\square \widehat{\mathbf{F}} = -\mathbf{V} + \mathbf{T} = -\mathbf{V} + \mathbf{e}_5 \mathbf{U} \quad (5.70)$$

Multiplying Eq. (5.69) on the right by \mathbf{F} and Eq. (5.70) on the left

$$\widehat{\mathbf{F}} \square \widehat{\mathbf{F}} = -\mathbf{FV} + \mathbf{FT} = -\mathbf{FV} + \mathbf{e}_5 \mathbf{FU} \quad (5.71)$$

$$\widehat{\mathbf{F}} \square \widehat{\mathbf{F}} = \mathbf{VF} + \mathbf{TF} = \mathbf{VF} + \mathbf{e}_5 \mathbf{UF} \quad (5.72)$$

Adding Eqs. (5.71) and (5.72)

$$\begin{aligned} \widehat{\mathbf{F}} \square \widehat{\mathbf{F}} + \widehat{\mathbf{F}} \square \widehat{\mathbf{F}} &= -\mathbf{FV} + \mathbf{e}_5 \mathbf{FU} + \mathbf{VF} + \mathbf{e}_5 \mathbf{UF} \\ &= -(\mathbf{FV} - \mathbf{VF}) + \mathbf{e}_5 (\mathbf{FU} + \mathbf{UF}) = -(\mathbf{FV} - \mathbf{VF}) + (\mathbf{Fe}_5 \mathbf{U} - \mathbf{e}_5 \mathbf{UF}) \end{aligned} \quad (5.73)$$

The first term $(\mathbf{FV} - \mathbf{VF})$, in Eq. (5.73), is the space-time Faraday law when $\mathbf{V} = \mu_0 c \rho_s \mathbf{e}_0 + \mathbf{j}_s$ or $\mathbf{V} = \mu_0 (c q_s \mathbf{e}_0 + q_s \mathbf{v}_s)$. The second term is zero.

We now evaluate $\mathbf{FV} - \mathbf{VF}$ in Eq. (5.73)

$$\mathbf{V} = \mathbf{e}_0 \operatorname{div} \frac{\mathbf{E}}{c} - \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t} + \operatorname{curl} \mathbf{B} \quad (5.74)$$

$$\mathbf{F} = \mathbf{e}_0 \mathbf{e}_1 \frac{E_x}{c} + \mathbf{e}_0 \mathbf{e}_2 \frac{E_y}{c} + \mathbf{e}_0 \mathbf{e}_3 \frac{E_z}{c} + \mathbf{e}_1 \mathbf{e}_2 B_z + \mathbf{e}_3 \mathbf{e}_1 B_y + \mathbf{e}_2 \mathbf{e}_3 B_x \quad (5.75)$$

\mathbf{F} is sometimes written

$$\begin{aligned}\mathbf{F} &= \mathbf{e}_0\mathbf{e}_1\left(\frac{E_x}{c} + \mathbf{e}_5B_x\right) + \mathbf{e}_0\mathbf{e}_2\left(\frac{E_y}{c} + \mathbf{e}_5B_y\right) + \mathbf{e}_0\mathbf{e}_3\left(\frac{E_z}{c} + \mathbf{e}_5B_z\right) \\ &= \frac{1}{c}\mathbf{E} + \mathbf{e}_5\mathbf{B}\end{aligned}$$

where \mathbf{E} and \mathbf{B} now both take the form of time-like bivectors.

$$\begin{aligned}\mathbf{V}\mathbf{F} - \mathbf{F}\mathbf{V} &= -\mathbf{e}_0\left[\left(-\frac{1}{c^2}\frac{\partial E_x}{\partial t} + (\text{curl } B)_x\right)\frac{E_x}{c}\right. \\ &\quad \left.-\left(-\frac{1}{c^2}\frac{\partial E_y}{\partial t} + (\text{curl } B)_y\right)\frac{E_y}{c} + \left(-\frac{1}{c^2}\frac{\partial E_z}{\partial t} + (\text{curl } B)_z\right)\frac{E_z}{c}\right] \\ &\quad +\mathbf{e}_1\left[\left(-\frac{1}{c^2}\frac{\partial E_z}{\partial t} + (\text{curl } B)_z\right)B_x\right. \\ &\quad \left.-\left(-\frac{1}{c^2}\frac{\partial E_y}{\partial t} + (\text{curl } B)_y\right)B_z + \left(-\text{div}\frac{\mathbf{E}}{c}\right)E_x\right] \\ &\quad +\mathbf{e}_2\left[\left(-\frac{1}{c^2}\frac{\partial E_x}{\partial t} + (\text{curl } B)_x\right)B_z\right. \\ &\quad \left.-\left(-\frac{1}{c^2}\frac{\partial E_z}{\partial t} + (\text{curl } B)_z\right)B_x + \left(-\text{div}\frac{\mathbf{E}}{c}\right)E_y\right] \\ &\quad +\mathbf{e}_3\left[\left(-\frac{1}{c^2}\frac{\partial E_x}{\partial t} + (\text{curl } B)_y\right)B_x\right. \\ &\quad \left.-\left(-\frac{1}{c^2}\frac{\partial E_x}{\partial t} + (\text{curl } B)_x\right)B_{sy} + \left(-\text{div}\frac{\mathbf{E}}{c}\right)E_z\right]\end{aligned}$$

Thus

$$\begin{aligned}\mathbf{V}\mathbf{F} - \mathbf{F}\mathbf{V} &= -(\mathbf{F}\mathbf{V} - \mathbf{V}\mathbf{F}) = \mathbf{e}_0\left(\frac{1}{c^2}\frac{\partial \mathbf{E}^2}{\partial t} - \frac{\mathbf{E}}{c} \cdot \text{curl } \mathbf{B}\right) \\ &\quad -\frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} \times \mathbf{B} + \mathbf{B} \times \text{curl } \mathbf{B} - \left(\text{div}\frac{\mathbf{E}}{c}\right)\mathbf{E}\end{aligned}\quad (5.76)$$

Now evaluate $\mathbf{e}_5(\mathbf{F}\mathbf{U} - \mathbf{U}\mathbf{F})$ in Eq. (5.73).

$$\begin{aligned}
\mathbf{T} &= \mathbf{e}_5\mathbf{U}, & \mathbf{U} &= -\mathbf{e}_5\mathbf{T} \\
\mathbf{T} &= \mathbf{e}_0\mathbf{e}_1\mathbf{e}_2 \left(-\frac{1}{c}(\operatorname{curl}\mathbf{E})_z - \frac{1}{c}\frac{\partial B_z}{\partial t} \right) + \mathbf{e}_0\mathbf{e}_3\mathbf{e}_1 \left(-\frac{1}{c}(\operatorname{curl}\mathbf{E})_y - \frac{\partial B_y}{c\partial t} \right) \\
&\quad + \mathbf{e}_0\mathbf{e}_2\mathbf{e}_3 \left(-\frac{1}{c}(\operatorname{curl}\mathbf{E})_x - \frac{1}{c}\frac{\partial B_x}{\partial t} \right) + \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\operatorname{div}\mathbf{B} \\
\mathbf{U} &= -\mathbf{e}_5\mathbf{T} = \mathbf{e}_3 \left(-\frac{1}{c}(\operatorname{curl}\mathbf{E})_z - \frac{1}{c}\frac{\partial B_z}{\partial t} \right) + \mathbf{e}_2 \left(-\frac{1}{c}(\operatorname{curl}\mathbf{E})_y - \frac{\partial B_y}{c\partial t} \right) \\
&\quad + \mathbf{e}_1 \left(-\frac{1}{c}(\operatorname{curl}\mathbf{E})_x - \frac{\partial B_x}{c\partial t} \right) + \mathbf{e}_0\operatorname{div}\mathbf{B} \\
&= \mathbf{e}_0u_0 + \mathbf{e}_1u_1 + \mathbf{e}_2u_2 + \mathbf{e}_3u_3 \\
\mathbf{F} &\equiv \frac{E_x}{c}\mathbf{e}_0\mathbf{e}_1 + \frac{E_y}{c}\mathbf{e}_0\mathbf{e}_2 + \frac{E_z}{2}\mathbf{e}_0\mathbf{e}_3 + B_x\mathbf{e}_1\mathbf{e}_2 + B_y\mathbf{e}_3\mathbf{e}_1 + B_z\mathbf{e}_2\mathbf{e}_3
\end{aligned}$$

$$\begin{aligned}
\mathbf{e}_0\mathbf{e}_1\mathbf{e}_2 &= \mathbf{e}_5\mathbf{e}_3, & \mathbf{e}_0\mathbf{e}_3\mathbf{e}_1 &= \mathbf{e}_5\mathbf{e}_2, & \mathbf{e}_0\mathbf{e}_2\mathbf{e}_3 &= \mathbf{e}_5\mathbf{e}_1, & \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 &= \mathbf{e}_5\mathbf{e}_0 \\
\mathbf{e}_5\mathbf{e}_0\mathbf{e}_1\mathbf{e}_2 &= -\mathbf{e}_3 & \mathbf{e}_3\mathbf{e}_0\mathbf{e}_3\mathbf{e}_1 &= -\mathbf{e}_2 & \mathbf{e}_5\mathbf{e}_0\mathbf{e}_2\mathbf{e}_3 &= -\mathbf{e}_1 & \mathbf{e}_5\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 &= -\mathbf{e}_0
\end{aligned}$$

$$\begin{aligned}
&\mathbf{e}_5(\mathbf{U}\mathbf{F} + \mathbf{F}\mathbf{U}) = \\
&\left[\frac{1}{c^2} \left((\operatorname{curl}\mathbf{E})_y + \frac{\partial B_y}{\partial t} \right) E_x - \frac{1}{c^2} \left((\operatorname{curl}\mathbf{E})_x + \frac{\partial B_x}{\partial t} \right) E_y - (\operatorname{div}\mathbf{B}) B_x \right] \mathbf{e}_3 \\
&+ \left[\frac{1}{c^2} \left((\operatorname{curl}\mathbf{E})_x + \frac{\partial B_x}{\partial t} \right) E_z - \frac{1}{c^2} \left((\operatorname{curl}\mathbf{E})_z + \frac{\partial B_z}{\partial t} \right) E_x - (\operatorname{div}\mathbf{B}) B_y \right] \mathbf{e}_2 \\
&+ \left[\frac{1}{c^2} \left((\operatorname{curl}\mathbf{E})_z + \frac{\partial B_z}{\partial t} \right) E_y - \frac{1}{c^2} \left((\operatorname{curl}\mathbf{E})_y + \frac{\partial B_y}{\partial t} \right) E_z - (\operatorname{div}\mathbf{B}) B_z \right] \mathbf{e}_1 \\
&+ \left[\frac{1}{c} \left((\operatorname{curl}\mathbf{E})_z + \frac{\partial B_z}{\partial t} \right) B_z + \frac{1}{c} \left((\operatorname{curl}\mathbf{E})_y + \frac{\partial B_y}{\partial t} \right) B_y \right. \\
&\quad \left. + \frac{1}{c} \left((\operatorname{curl}\mathbf{E})_x + \frac{\partial B_x}{\partial t} \right) B_x \right] \mathbf{e}_0 \\
&= \mathbf{e}_0 \left[\frac{1}{c}(\operatorname{curl}\mathbf{E}) \cdot \mathbf{B} + \frac{\partial \mathbf{B}^2}{c\partial t} \frac{1}{2} \right] - \frac{1}{c^2}\mathbf{E} \times \operatorname{curl}\mathbf{E} - \frac{1}{c^2}\mathbf{E} \times \frac{\partial \mathbf{B}}{\partial t} - (\operatorname{div}\mathbf{B})\mathbf{B} \quad (5.77)
\end{aligned}$$

Add Eqs. (5.76) and (5.77) so that

$$\begin{aligned}
& \hat{\mathbf{F}}\square\hat{\mathbf{F}} + \mathbf{F}\square\mathbf{F} \\
&= (\mathbf{V}\mathbf{F} - \mathbf{F}\mathbf{V}) + \underline{\mathbf{e}_5(\mathbf{U}\mathbf{F} + \mathbf{F}\mathbf{U})} \\
&= \mathbf{e}_0 \left(\frac{\text{IIIa}}{c^3} \frac{\partial \mathbf{E}^2}{\partial t} \frac{1}{2} - \frac{\mathbf{E}}{c} \cdot \text{curl } \mathbf{B} \right) - \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t} \times \mathbf{B} + \mathbf{B} \times \text{curl } \mathbf{B} - \left(\text{div} \frac{\mathbf{E}}{c} \right) \mathbf{E} \\
&+ \mathbf{e}_0 \left(\frac{\text{IIIb}}{c \partial t} \frac{\mathbf{B}^2}{2} + \underline{\mathbf{B} \cdot \text{curl} \frac{\mathbf{E}}{c}} \right) - \frac{1}{c^2} \mathbf{E} \times \frac{\partial \mathbf{B}}{\partial t} - \frac{1}{c^2} \underline{(\mathbf{E} \times \text{curl } \mathbf{E})} - \underline{(\text{div } \mathbf{B}) \mathbf{B}} \quad (5.78) \\
&= \mathbf{e}_0 \left[\frac{\partial}{c \partial t} \left(\frac{\mathbf{E}^2}{c^2} + \mathbf{B}^2 \right) + \frac{1}{c} (\underline{\mathbf{B} \cdot \text{curl } \mathbf{E}} - \mathbf{E} \cdot \text{curl } \mathbf{B}) \right] \\
&\quad - \frac{1}{c^2} \frac{\partial}{\partial t} \underline{(\mathbf{E} \times \mathbf{B})} + \frac{1}{c^2} \left[\underline{(\nabla \times \mathbf{E}) \times \mathbf{E}} - (\text{div } \mathbf{E}) \mathbf{E} \right] \\
&\quad - \left[\underline{(\nabla \times \mathbf{B}) \times \mathbf{B}} - \underline{(\text{div } \mathbf{B}) \mathbf{B}} \right] \quad (5.79)
\end{aligned}$$

In Eq. (5.78) the terms underlined come from $\mathbf{e}_5(\mathbf{U}\mathbf{F} + \mathbf{F}\mathbf{U})$. The others from $\mathbf{V}\mathbf{B} - \mathbf{B}\mathbf{V}$. The double underline involves both groups of terms.

Using the vector identity

$$\text{div } \mathbf{A} \times \mathbf{B} = \mathbf{B} \cdot \text{curl } \mathbf{A} - \mathbf{A} \cdot \text{curl } \mathbf{B}$$

one can write

$$\text{div } \mathbf{E} \times \mathbf{B} = \mathbf{B} \cdot \text{curl } \mathbf{E} - \mathbf{E} \cdot \text{curl } \mathbf{B}$$

$$\begin{aligned}
\hat{\mathbf{F}}\square\hat{\mathbf{F}} + \mathbf{F}\square\mathbf{F} &= \mathbf{e}_0 \left[\frac{\partial}{c \partial t} \left(\frac{\mathbf{E}^2}{c^2} + \mathbf{B}^2 \right) + \frac{1}{c} \underline{\underline{\nabla \cdot (\mathbf{E} \times \mathbf{B})}} \right] \\
&\quad - \frac{\partial}{\partial t} \frac{\underline{(\mathbf{E} \times \mathbf{B})}}{c^2} + \frac{1}{c^2} \left[\underline{(\nabla \times \mathbf{E}) \times \mathbf{E}} - \mathbf{E} \underline{\nabla \cdot \mathbf{E}} \right] \\
&\quad + \left[\underline{(\nabla \times \mathbf{B}) \times \mathbf{B}} - \underline{\mathbf{B} \underline{\nabla \cdot \mathbf{B}}} \right] = \rho \mathbf{E} + \frac{\mathbf{J} \times \mathbf{B}}{c} \quad (5.80)
\end{aligned}$$

This is the same as Jackson (1975, 238)

Term I in Eq. (5.80) is the sum of Terms Ia and Ib in Eq. (5.78). That is, it is a mix of a contribution from the first set of Maxwell's equations, namely $-\left[(\partial\mathbf{E}/\partial t) \times \mathbf{B}\right]/c^2$, and the second set of Maxwell's equations, namely $[\mathbf{E} \times (\partial\mathbf{B}/\partial t)]/c^2$.

Similarly Term II in Eq. (5.80), namely $\nabla \cdot (\mathbf{E} \times \mathbf{B})/c = (\mathbf{E} \operatorname{curl} \mathbf{B} - \mathbf{B} \operatorname{curl} \mathbf{E})/c$ is the sum of Terms IIa and IIb in Eq. (5.78). Term III in Eq. (5.80) is the sum of IIIa and IIIb in Eq. (5.78). The former comes from the first set of Maxwell's equations while IIIb comes from the second set of Maxwell's equations.

Term IV in Eq. (5.80) comes from Maxwell's second set and Term V from the first set.

Term VI comes from Maxwell's first set and Term VII from the second set.

In summary, the underlined Terms in Eq. (5.80) come from the second set of Maxwell's equations.

The non-underlined come from the first set of Maxwell's equations.

The double underlined are a combination of terms from both sets of Maxwell's equations.

Half of the terms in Eqs. (5.78) and (5.79) come from $\mathbf{e}_5(\mathbf{UB} + \mathbf{BU})$. It gives rise to terms

$$-\frac{1}{c^2}\mathbf{E} \times \left(\operatorname{curl} \mathbf{E} + \frac{\partial\mathbf{B}}{\partial t}\right) \quad \text{and} \quad \mathbf{B}(\operatorname{div} \mathbf{B}) \quad (5.81)$$

From Maxwell's second set of equations

$$\operatorname{curl} \mathbf{E} + \frac{\partial\mathbf{B}}{\partial t} = 0 \quad (5.82)$$

$$\operatorname{div} \mathbf{B} = 0 \quad (5.83)$$

In Eq. (5.78) the coefficient of \mathbf{e}_0 is

$$\mathbf{e}_0 \left[\mathbf{B} \cdot \left(\operatorname{curl} \mathbf{E} + \frac{\partial\mathbf{B}}{\partial t} \right) \right]$$

By Maxwell's Eq. (5.82) this term is zero.

$$\begin{array}{ll} \widehat{\mathbf{F}}\square &= \operatorname{Vec} + \operatorname{Triv} & \widehat{\mathbf{F}}\square\mathbf{F} &= \mathbf{VF} + \mathbf{TF} \\ \widehat{\mathbf{F}}\square &= -\operatorname{Vec} + \operatorname{Triv} & \mathbf{F}\square\widehat{\mathbf{F}} &= -\mathbf{FV} + \mathbf{FT} \\ \mathbf{F}\square - \square\mathbf{F} &= \operatorname{Vec} & \widehat{\mathbf{F}}\square\mathbf{F} + \mathbf{F}\square\widehat{\mathbf{F}} &= -(\mathbf{FV} + \mathbf{VF}) + (\mathbf{FT} + \mathbf{TF}) \\ \mathbf{F}\square + \square\mathbf{F} &= \operatorname{Triv} & \mathbf{F}\square\widehat{\mathbf{F}} + \widehat{\mathbf{F}}\square\mathbf{F} &= -(\mathbf{FV} - \mathbf{VF}) + (\mathbf{FT} + \mathbf{TF}) \end{array}$$

$$\mathbf{F}\square = \mathbf{V} + \mathbf{T}$$

$$\widehat{\square}\mathbf{F} = -\mathbf{V} + \mathbf{T} = -\mathbf{V} + \mathbf{e}_5\mathbf{U}$$

$$\widehat{\mathbf{F}}\square = \mathbf{V} + \mathbf{T} = \mathbf{V} + \mathbf{e}_5\mathbf{U}$$

$$\mathbf{F}\widehat{\square}\mathbf{F} = -\mathbf{FV} + \mathbf{FT} = -\mathbf{FV} + \mathbf{Fe}_5\mathbf{U} = -\mathbf{FV} + \mathbf{e}_5\mathbf{FU}$$

$$\widehat{\mathbf{F}}\square\mathbf{F} = \mathbf{VF} + \mathbf{TF} = \mathbf{VF} + \mathbf{e}_5\mathbf{UF} = \mathbf{VF} + \mathbf{e}_5\mathbf{UF}$$

$$\mathbf{F}\widehat{\square}\mathbf{F} + \widehat{\mathbf{F}}\square\mathbf{F} = -(\mathbf{FV} - \mathbf{VF}) + \mathbf{e}_5(\mathbf{FU} + \mathbf{UF})$$

