4
SPACE-INVARIANT. SPACE-TIME INTERVAL INVARIANT

In this chapter, space-time algebra is used to obtain the Lorentz transformations and other results in special relativity. The starting point is to employ the invariant of the space-time interval as an ansatz.

4.1 Some Applications of Space-time Formalism to Special Relativity. Space-time Interval Invariant as an Ansatz Leading to the Lorentz Transformations

We first describe a procedure for rotating vectors. Doing so also leads to a few standard trigonometric identities.

Rotation counterclockwise of the triangle OAB in Fig. 4.1a to triangle $OA'B'$ in Fig. 4.1b can be accomplished by rotating $\mathbf{r}_1$ through an angle θ and then rotating $\mathbf{r}_2$ through an angle θ. The lengths remain constant so that r_1^2 and r_2^2 remain constant.

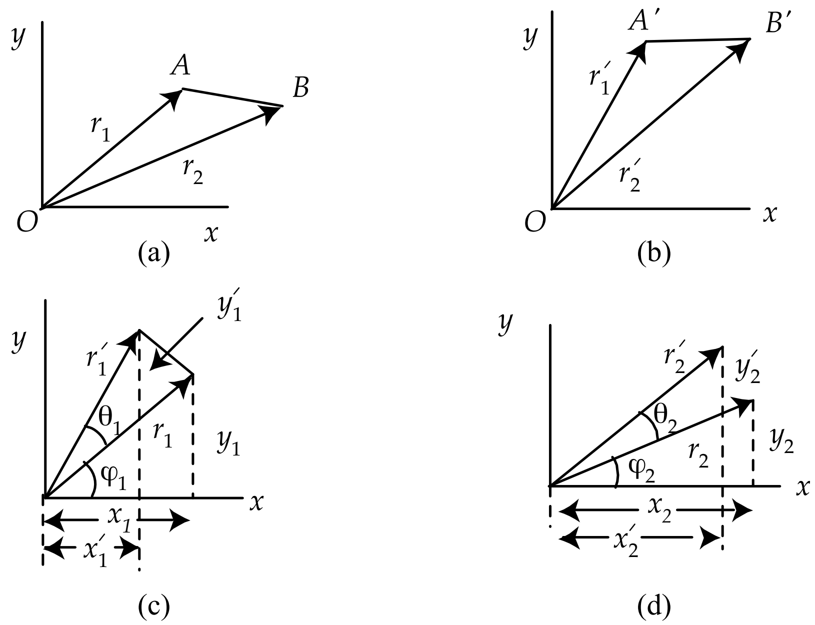

Fig. 4.1

Before rotation : $\mathbf{r}_1 = \mathbf{e}_1 x_1 + \mathbf{e}_2 y_1$ After rotation : $\mathbf{r}_1' = \mathbf{e}_1 x_1' + \mathbf{e}_2 y_1'$

The counterclockwise rotation of a vector $\mathbf{r}_1$ through an angle θ_1, labeled $\mathbf{r}_1'$ in its new orientation, may be achieved by performing the following bilateral operation on $\mathbf{r}_1$.

$$\begin{aligned}\mathbf{r}_1' &= x_1'\mathbf{e}_1 + y_1'\mathbf{e}_2 = \mathbf{e}^{-\mathbf{e}_1\mathbf{e}_2\theta_1/2}\mathbf{r}_1\mathbf{e}^{\mathbf{e}_1\mathbf{e}_2\theta_1/2} \\ &= \mathbf{e}^{-\mathbf{e}_1\mathbf{e}_2\theta_1/2}\left(\mathbf{e}_1 x_1 + \mathbf{e}_2 y_1\right)\mathbf{e}^{\mathbf{e}_1\mathbf{e}_2\theta_1/2} \qquad (4.1)\end{aligned}$$

The bilateral operation on $\mathbf{e}_1 x_1$ may be converted to a unilateral operation as follows. The same result applies to $\mathbf{e}_2 y_1$.

$$\begin{aligned}\mathbf{e}_1 x_1 \mathbf{e}^{\mathbf{e}_1\mathbf{e}_2\theta_1/2} &= \mathbf{e}_1 x_1\left(\cos\theta_1/2 + \mathbf{e}_1\mathbf{e}_2\sin\theta_1/2\right) \\ &= \left(\cos\theta_1/2 - \mathbf{e}_1\mathbf{e}_2\sin\theta_1/2\right)\mathbf{e}_1 x_1 = \mathbf{e}^{-\mathbf{e}_1\mathbf{e}_2\theta/2}\mathbf{e}_1 x_1\end{aligned}$$

$$\begin{aligned}\mathbf{e}_2 y_1 \mathbf{e}^{\mathbf{e}_1\mathbf{e}_2\theta_1/2} &= \mathbf{e}_2 y_1\left(\cos\theta_1/2 + \mathbf{e}_1\mathbf{e}_2\sin\theta_1/2\right) \\ &= \left(\cos\theta_1/2 - \mathbf{e}_1\mathbf{e}_2\sin\theta_1/2\right)\mathbf{e}_2 y_1 = \mathbf{e}^{\mathbf{e}_1\mathbf{e}_2\theta_1/2}\mathbf{e}_2 y_1\end{aligned}$$

$$\begin{aligned}r_1' &= x_1'\mathbf{e}_1 + y_1'\mathbf{e}_2 = \mathbf{e}^{-\mathbf{e}_1\mathbf{e}_2\theta_1/2}\mathbf{e}^{-\mathbf{e}_1\mathbf{e}_2\theta_1/2}\left(\mathbf{e}_1 x_1 + \mathbf{e}_2 y_2\right) \\ &= \mathbf{e}^{-\mathbf{e}_1\mathbf{e}_2\theta_1}\left(\mathbf{e}_1 x_1 + \mathbf{e}_2 y_1\right) \qquad (4.2)\end{aligned}$$

Therefore the bilateral operation involving $\theta_1/2$ becomes a unilateral operation involving θ_1. Thus by Eq. (4.2),

$$\begin{aligned}
\mathbf{e}_1 x_1' + \mathbf{e}_2 y_1' &= (\cos\theta_1 - \mathbf{e}_1\mathbf{e}_2 \sin\theta_1)(\mathbf{e}_1 x_1 + \mathbf{e}_2 y_1) \\
&= \mathbf{e}_1 x_1 \cos\theta_1 + \mathbf{e}_2 y \cos\theta_1 + \mathbf{e}_2 x_1 \sin\theta_1 - \mathbf{e}_1 y_1 \sin\theta_1 \\
\mathbf{e}_1 x_1' + \mathbf{e}_2 y_1' &= \mathbf{e}_1 (x_1 \cos\theta_1 - y_1 \sin\theta_1) + \mathbf{e}_2 (x_1 \sin\theta_1 + y_1 \cos\theta_1) \qquad (4.3)
\end{aligned}$$

Equating coefficients of $\mathbf{e}_1$ and $\mathbf{e}_2$,

$$x_1' = x_1 \cos\theta_1 - y_1 \sin\theta_1 \qquad (4.4)$$

$$y_1' = x_1 \sin\theta_1 + y_1 \cos\theta_1 \qquad (4.5)$$

In Eqs. (4.4), (4.5) the coordinates of $\mathbf{r}_1$ after rotation are expressed in terms of its coordinates before rotation. We require the length of $\mathbf{r}_1$ to be unchanged by the rotation, that is

$$x_1'^2 + y_1'^2 = x_1^2 + y_1^2$$

This condition is satisfied by Eq. (4.3) since

$$\begin{aligned}
x_1'^2 + y_1'^2 &= (x_1 \cos\theta_1 - y_1 \sin\theta_1)^2 + (x_1 \sin\theta_1 + y_1 \cos\theta_1)^2 \\
&= x_1^2 \left(\sin^2\theta_1 + \cos^2\theta_1\right) + y_1^2 \left(\sin^2\theta_1 + \cos^2\theta_1\right) = x_1^2 + y_1^2
\end{aligned}$$

If x_1 and y_1 in Eq. (4.4) are given by

$$x_1 = r_1 \cos\varphi \qquad y_1 = r_1 \sin\varphi \quad \text{and} \quad x_1' = r_1' \cos(\theta_1 + \varphi) \qquad y_1' = r_1' \sin(\theta_1 + \varphi)$$

Then

$$\begin{aligned}
r_1' \cos(\theta_1 + \varphi_1) &= r_1 (\cos\varphi_1 \cos\theta_1 - \sin\varphi_1 \sin\theta_1) \\
r_1' \sin(\theta_1 + \varphi_1) &= r_1 (\cos\varphi_1 \sin\theta_1 + \sin\varphi_1 \cos\theta_1)
\end{aligned}$$

Since $r_1' = r_1$ and $r_2' = r_2$ we obtain the trigonometric formulas:

$$\begin{aligned}
\cos(\theta_1 + \varphi) &= \cos\varphi_1 \cos\theta_1 - \sin\varphi_1 \sin\theta_1 \\
\sin(\theta_1 + \varphi) &= \cos\varphi_1 \sin\theta_1 + \sin\varphi_1 \cos\theta_1
\end{aligned}$$

Note that r_1 and r_1' may be written in terms of components without initially referring to a figure by proceeding as follows. Let us say that we are looking for quantities p_1 and q_1 such that

$$x_1 = r_1 p_1 \qquad y_1 = r_1 q_1 \qquad (4.6)$$

where $r_1 p_1$ locates coordinate x_1 and $r_1 q_1$ locates coordinate y_1. $r_1 =$ constant.

$$\begin{array}{lrcl} \text{Then} & \mathbf{r}_1 & = & \mathbf{e}_1 x_1 + \mathbf{e}_2 y_1 = \mathbf{e}_1 r_1 p_1 + \mathbf{e}_2 r_1 q_1 \\ \text{Likewise} & \mathbf{r}'_1 & = & \mathbf{e}_1 x'_1 + \mathbf{e}_2 y'_1 = \mathbf{e}_1 r'_1 p'_1 + \mathbf{e}_2 r'_2 q'_1 \end{array}$$

where we require $\mathbf{r}_1 = \mathbf{r}'_1$ so

$$r_1^2 = r_1'^2 = x_1^2 + y_1^2 = x_1'^2 + y_1'^2 = \text{const.} \tag{4.7}$$

the parameters must then have the property

$$r_1^2 p_1^2 + r_1^2 q_1^2 = r_1^2 \qquad r_1'^2 p_1'^2 + r_1'^2 q_1'^2 = r_1'^2 \tag{4.8}$$

$$p_1^2 + q_1^2 = 1 \qquad p_1'^2 + q_1'^2 = 1 \tag{4.9}$$

The sine and cosine of an angle θ_1 have this property

$$\sin^2 \theta_1 + \cos^2 \theta_1 = 1$$

So the parameters p_1 and q_1 can be identified with $\sin \theta_1$ and $\cos \theta_1$, and the definition of p_1 and q_1 reduce to functions of a single parameter θ_1 so that one can now write

$$\begin{array}{lrcl} & \mathbf{r}_1 & = & \mathbf{e}_1 r_1 \cos \theta_1 + e_2 r_1 \sin \theta_1 = \mathbf{e}_1 x_1 + \mathbf{e}_2 y_1 \\ \text{and} & \mathbf{r}'_1 & = & \mathbf{e}_1 r_1 \cos \theta'_1 + \mathbf{e}_2 r_1 \sin \theta'_1 = \mathbf{e}_1 x'_1 + \mathbf{e}_2 y'_2 \end{array}$$

The sine and cosine relate directly to the physical geometry so that going from x_1, y_1 to x'_1, y'_1 is simply a two dimensional rotation with length invariant. The same procedure may be applied to $\mathbf{r}_2$.

The same procedure is now applied to space-time intervals to obtain the Lorentz transformations for space and time coordinates.

4.2 Space-Time Interval. Space-Time "Rotation"

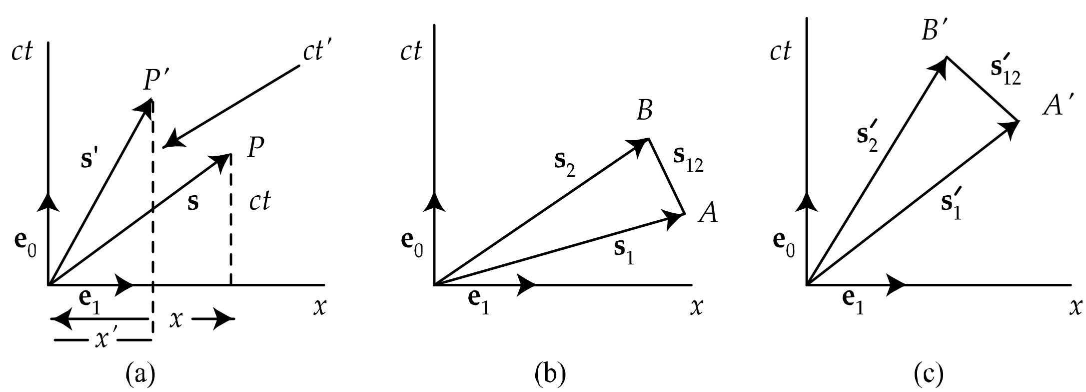

Fig. 4.2

Given: Two "observers" S and S' drifting in space along a straight line connecting them. They agree on two things: Their relative velocity is $\mathtt{w}$, and when each measures the velocity of light they obtain the same value c. Let each observer record his time and space values of two events. To simplify the algebra, consider the events as taking place at different times and positions along a straight line labeled the x-axis. Time is referred to an axis orthogonal to the space axis. For simplicity we allow one event to take place when the space and time origins of each coincide. Both observers assign a value of 0 to the time and distance of the event when their origins coincide. They both observe a second event and record their respective time and space values along orthogonal axes defined by the unit vector $\mathbf{e}_0$ for time and $\mathbf{e}_1$ for space. An event is shown at P and another at P' in Fig. 4.2a. Fig. 4.2a shows the "separation" $\mathbf{s}$ of the event P from the origin as measured by S. Also shown is the separation $\mathbf{s}'$ of the same event from the event at the origin as measured by S'. The separation of the two events, that is, the coordinates of the second event as measured by S is given by

$$\mathbf{s} = \mathbf{e}_0 ct + \mathbf{e}_1 x \quad \text{or} \quad d\mathbf{s} = \mathbf{e}_0 cdt + \mathbf{e}_1 dx \tag{4.10}$$

As measured by S' the second event coordinates are

$$\mathbf{s}' = \mathbf{e}_0 ct' + \mathbf{e}_1 x' \qquad d\mathbf{s}' = \mathbf{e}_0 cdt' + \mathbf{e}_1 dx' \tag{4.11}$$

We now proceed as we did for the space rotation of a vector. For a space-time vector, however, the square of the separation of the two events, also called the interval, as measured by S, Eq. (4.10) is:

$$s^2 = -(ct)^2 + x^2 \quad \text{or} \quad (ds)^2 = -(cdt)^2 + (dx)^2 \tag{4.12}$$

As measured by S', Eq. (4.11) the square of the separation is:

$$s'^2 = -(ct')^2 + x'^2 \qquad (ds')^2 = -(cdt')^2 + (dx')^2 \tag{4.13}$$

Special relativity requires that the interval as measured by both observers have the same value for each, so that

$$s^2 = s'^2 \quad \text{or} \quad ds^2 = ds'^2 \tag{4.14}$$

Analogous to the rotation of two vectors in three space, we now look for a parameter p that relates s to ct and a parameter q that relates s to x, that is

$$ct = sp \qquad x = sq \quad \text{or} \quad cdt = (ds)\,p \qquad dx = (ds)\,q \tag{4.15}$$

Putting Eq. (4.15) into (4.12), the parameters must then have the property

$$-s^2p^2 + s^2q^2 = s^2 \quad \text{or} \quad -ds^2p^2 + ds^2q^2 = ds^2$$

Therefore, from either form, $-p^2 + q^2 = 1.$

This is satisfied by[1]

$$p = \sinh u, \qquad q = \cosh u$$

since

$$-\sinh^2 u + \cosh^2 u = 1$$

The two parameters p and q have now been replaced by the function of a single parameter u. Then analogous to Eq. (4.1) we write

$$\mathbf{s}' = (\mathbf{e}_0 ct' + \mathbf{e}_1 x') = \mathbf{e}^{-\mathbf{e}_0\mathbf{e}_1 u/2}\mathbf{s}\mathbf{e}^{\mathbf{e}_0\mathbf{e}_1 u/2} = \mathbf{e}^{-\mathbf{e}_0\mathbf{e}_1 u/2}(ct\mathbf{e}_0 + x\mathbf{e}_1)\,\mathbf{e}^{\mathbf{e}_0\mathbf{e}_1 u/2} \tag{4.16}$$

Carrying out the details

$$\begin{aligned}\mathbf{e}_1\mathbf{e}^{\mathbf{e}_0\mathbf{e}_1 u/2} &= \mathbf{e}_1(\cosh u/2 + \mathbf{e}_0\mathbf{e}_1 \sinh u/2) = (\cosh u/2 - \mathbf{e}_0\mathbf{e}_1\sinh u/2)\,\mathbf{e}_1 \\ &= \mathbf{e}^{-\mathbf{e}_0\mathbf{e}_1 u/2}\mathbf{e}_1\end{aligned}$$

Likewise

$$\begin{aligned}\mathbf{e}_0\mathbf{e}^{\mathbf{e}_0\mathbf{e}_1 u/2} &= \mathbf{e}_0(\cosh u/2 + \mathbf{e}_0\mathbf{e}_1 \sinh u/2) = (\cosh u/2 - \mathbf{e}_0\mathbf{e}_1\sinh u/2)\,\mathbf{e}_0 \\ &= \mathbf{e}^{-\mathbf{e}_0\mathbf{e}_1 u/2}\mathbf{e}_0 \\ \text{Thus}\quad \mathbf{s}' &= \mathbf{e}_0 ct' + \mathbf{e}_1 x' = \mathbf{e}^{-\mathbf{e}_0\mathbf{e}_1 u}(\mathbf{e}_0 ct + \mathbf{e}_1 x) \qquad (4.17)\end{aligned}$$

where the bilateral operation has been replaced by the single term $\mathbf{e}^{-\mathbf{e}_0\mathbf{e}_1 u}$ acting on the left. Writing out the exponential on the right hand side of the above

$$\begin{aligned}\mathbf{s}' &= \mathbf{e}_0 ct' + \mathbf{e}_1 x' = (\cosh u - \mathbf{e}_0\mathbf{e}_1 \sinh u)\,\mathbf{e}_0 ct + (\cosh u - \mathbf{e}_0\mathbf{e}_1\sinh u)\,\mathbf{e}_1 x \\ &= \mathbf{e}_0(ct\cosh u - x\sinh u) + \mathbf{e}_1(-ct\sinh u + x\cosh u) \qquad (4.18)\end{aligned}$$

Equating coefficients of $\mathbf{e}_1$ and $\mathbf{e}_0$

$$x' = x\cosh u - ct\sinh u \tag{4.19}$$

$$ct' = -x\sinh u + ct\cosh u \tag{4.20}$$

[1] The hyperbolic sine and cosine are defined by

$\sinh\theta = \dfrac{\mathbf{e}^{\theta} - \mathbf{e}^{-\theta}}{2} = -\sinh(-\theta) \qquad \cosh\theta = \dfrac{\mathbf{e}^{\theta} + \mathbf{e}^{-\theta}}{2} = \cosh(-\theta) \qquad \cosh^2\theta - \sinh^2\theta = 1$

$\tanh\theta = \dfrac{\mathbf{e}^{\theta} - \mathbf{e}^{-\theta}}{\mathbf{e}^{\theta} + \mathbf{e}^{-\theta}} = \dfrac{\mathbf{e}^{2\theta} - 1}{\mathbf{e}^{2\theta} + 1} = -\tanh(-\theta) = \dfrac{\sinh\theta}{\cosh\theta} \qquad \sinh\theta = \tanh\theta\cosh\theta$

$\sinh\theta = \tanh\theta/\sqrt{1 - \tanh^2\theta} \qquad \cosh\theta = 1/\sqrt{1 - \tanh^2\theta}$

For purely imaginary arguments $\sinh(i\theta) = i\sin\theta \qquad \cosh(i\theta) = \cos\theta$

To obtain the inverse, evaluate:

$$
\begin{aligned}
\mathbf{s} &= \mathbf{e}_0 ct + \mathbf{e}_1 x = \mathbf{e}^{\mathbf{e}_0\mathbf{e}_1 u}\left(\mathbf{e}_0 ct' + \mathbf{e}_1 x'\right) \\
&= \left(\cosh u + \mathbf{e}_0\mathbf{e}_1 \sinh u\right)\left(\mathbf{e}_0 ct' + \mathbf{e}_1 x'\right) \\
\mathbf{s} &= \left(\cosh u + \mathbf{e}_0\mathbf{e}_1 \sinh u\right)\mathbf{e}_0 ct' + \left(\cosh u + \mathbf{e}_0\mathbf{e}_1 \sinh u\right)\mathbf{e}_1 x' \\
\mathbf{e}_0 ct + \mathbf{e}_1 x &= \mathbf{e}_0\left(ct'\cosh u + x'\sinh u\right) + \mathbf{e}_1\left(ct'\sinh u + x'\cosh u\right)
\end{aligned}
$$

Equating coefficients of $\mathbf{e}_1$ and $\mathbf{e}_0$

$$x = x'\cosh u + ct'\sinh u \tag{4.21}$$

$$ct = x'\sinh u + ct'\cosh u \tag{4.22}$$

Thus to invert Eqs. (4.19), (4.20), replace u by $-u$.

The above leaves the interval invariant since

$$
\begin{aligned}
x'^2 - ct'^2 &= \left(x\cosh u - ct\sinh u\right)^2 - \left(-x\sinh u + ct\cosh u\right)^2 \\
x'^2 - c^2t'^2 &= x^2\cosh^2 u + (ct)^2\sinh^2 u - x^2\sinh^2 u - (ct)^2\cosh^2 u \\
&= x^2\left(\cosh^2 u - \sinh^2 u\right) - (ct)^2\left(\cosh^2 u - \sinh^2 u\right) \\
x'^2 - c^2t'^2 &= x^2 - c^2t^2
\end{aligned}
\tag{4.23}
$$

At this stage the only physical input has been the velocity of light. Einstein states that the velocity of light is constant in describing the interval between two events. We have already introduced this by using the same c for both intervals, $\mathbf{s}$ and $\mathbf{s}'$. We still need to specify the parameter u. It must be a measure for which the two observers obtain the same value. Other than the velocity of light, they both agree that their velocity of separation is $\mathtt{w}$. The velocity of separation will be used to assign a value to $\sinh u$ and $\cosh u$. To do so consider the position of the origin of the frame S' as measured by the observer in frame S, Eqs. (4.21),(4.22) for $x' = 0$.

$$x = ct'\sinh u$$

Also $$ct = ct'\cosh u$$

Dividing $$\frac{x}{ct} = \tanh u$$

$\frac{x}{t}$ is clearly the velocity $\mathtt{w}$ of frame S' relative to S so

$$\frac{\mathtt{w}}{c} = \tanh u$$

From this (see footnote 1):

$$\sinh u = \frac{\mathtt{w}/c}{\sqrt{1-\mathtt{w}^2/c^2}}, \qquad \cosh u = \frac{1}{\sqrt{1-\mathtt{w}^2/c^2}} \tag{4.24}$$

Substituting into Eqs. (4.21) and (4.22), we obtain

$$x = \frac{x' + \mathtt{w}t'}{\sqrt{1-\mathtt{w}^2/c^2}}, \qquad y = y', \qquad z = z', \qquad t = \frac{t' + \mathtt{w}x'/c^2}{\sqrt{1-\mathtt{w}^2/c^2}} \tag{4.25}$$

This is the Lorentz transformation. Let $\gamma = 1/\sqrt{1-\mathtt{w}^2/c^2}$.

To express x', t' in terms of x, t simply change $\mathtt{w}$ to $-\mathtt{w}$ so that[2]

$$\begin{aligned} x' &= \gamma\,(x - \mathtt{w}t) = \gamma\,(x - \beta ct)\,, \qquad \beta = \mathtt{w}/c, \\ y' &= y, \qquad z' = z \\ t' &= \gamma\left(t - \frac{\mathtt{w}x}{c^2}\right) \rightarrow \; ct' = \gamma\left(ct - \frac{\mathtt{w}x}{c}\right) = \gamma\,(ct - \beta x) \end{aligned} \tag{4.26}$$

Thus the Lorentz transformation can be effected by the operation Eq. (4.16) or Eq. (4.17) where $\tanh u = \mathtt{w}/c$.

4.3 Acceleration and Force in Special Relativity

The interval in Clifford Algebra (space-time interval) is

$$d\mathbf{s} = \mathbf{e}_0 cdt + \mathbf{e}_1 dx + \mathbf{e}_2 dy + \mathbf{e}_3 dz = \mathbf{e}_0 cdt + d\mathbf{x} \tag{4.27}$$

ds^2 is an absolute invariant with c the velocity light constant. We regard it as invariant by fiat.

$$ds^2 = -c^2\,(dt)^2 + (d\mathbf{r}\cdot d\mathbf{r})$$

$d\mathbf{s}$ defines the "separation" of two infinitesimally close events in space-time. By equating $d\mathbf{s}$ to its value when $d\mathbf{x} = 0$, that is when the two events occur at the same point in space, at the origin in this case, but occur at times differing by $d\tau$ in which case its value is also given by $\mathbf{e}_0 cd\tau$, so that

$$d\mathbf{s} = \mathbf{e}_0 cdt + d\mathbf{x} = \mathbf{e}_0 cd\tau \tag{4.28}$$

[2] For observer in frame $\mathbf{S}'$

$x' = -ct\sinh u \qquad \frac{x'}{ct'} = -\frac{\sinh u}{\cosh u} = -\tanh u$

$ct' = ct\cosh u \qquad \frac{x'}{t'} = -\mathtt{w}$

$d\tau$ is an invariant since $d\mathbf{s}$ is an invariant.

$$\begin{aligned} d\mathbf{s} &= dt\left(\mathbf{e}_0 c + \mathbf{e}_1 v_x + \mathbf{e}_2 v_y + \mathbf{e}_3 v_z\right) = \mathbf{e}_0 c d\tau \\ ds^2 &= dt^2\left(-c^2 + v^2\right) = -c^2 d\tau^2 \end{aligned} \tag{4.29}$$

$$d\tau = dt\sqrt{1 - v^2/c^2} \tag{4.30}$$

Thus to change non-relativistic time dt to relativistic time, replace dt by

$$dt\sqrt{1 - v^2/c^2} = \frac{1}{\gamma_v} dt \qquad \gamma_v = \frac{1}{\sqrt{1 - v^2/c^2}}$$

From Eq. (4.27)

$$\begin{aligned} (-ds)^2 &= -c^2 dt^2 + dx^2 \\ &= -c^2 dt^2 \left[1 - \frac{1}{c^2}\left(\frac{dx}{dt}\right)^2\right] \\ &= -c^2 dt^2 \left(1 - \frac{v^2}{c^2}\right) \\ ds &= cdt \\ (-ds)^2 &= -c^2 dt^2 + dx^2 = -c^2 dt^2 \left[1 - \frac{1}{c^2}\left(\frac{dx}{dt}\right)^2\right] \end{aligned}$$

Taking positive sign

$$-ds = -c^2 dt^2 \left[1 - \frac{1}{c^2}\left(\frac{dx}{dt}\right)^2\right]^2$$

In summary, let $ds^2/c^2 = d\tau^2$

$$\begin{aligned} d\tau^2 &= -dt^2\left(1 - \frac{v^2}{c^2}\right) \rightarrow \quad \frac{dt}{d\tau} = \frac{\pm 1}{\left(1 - v^2/c^2\right)^{1/2}} = \frac{1}{\left(1 - v^2/c^2\right)^{1/2}} = \gamma_v \\ d\tau^2 &= \frac{dt^2}{\gamma^2} \qquad dt = \gamma d\tau = \frac{d\tau}{\left(1 - v^2/c^2\right)^{1/2}} \qquad \gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \end{aligned}$$

where we have taken the positive sign

$$\begin{aligned} ds &= cdt\left(1 - v^2/c^2\right)^{1/2} \\ \frac{d}{ds} &= \frac{1}{c\left(1 - v^2/c^2\right)^{1/2}} \frac{d}{dt} = \frac{\gamma}{c} \frac{d}{dt} \end{aligned}$$

or when differentiating with respect to t, replace by

$$\frac{d}{dt} \to \gamma \frac{d}{dt} = \frac{1}{\sqrt{1 - v^2/c^2}} \frac{d}{dt} \qquad v^2 = v_x^2 + v_y^2 + v_z^2$$

in order to make d/dt invariant. Also when forming the differential dt, replace by

$$dt \to \sqrt{1 - v^2/c^2} dt$$

in order to make it invariant. That is, $\sqrt{1 - v^2/c^2} dt$ is invariant, abbreviated $d\tau$ above. We will not always use the symbol $d\tau$ but will simply use $\sqrt{1 - v^2/c^2} dt$ in place of dt and write

$$\frac{1}{\sqrt{1 - v^2/c}} \frac{d}{dt} = \gamma \frac{d}{dt}$$

for the derivative with respect to t. $(ds)^2$ is

$$ds^2 = dx^2 + dy^2 + dz^2 - c^2 dt^2 = \text{invariant} \tag{4.31}$$

The usual procedure for showing that ds^2, the square of the interval, is invariant is to derive it from the Lorentz transformations. We postulate Eq. (4.31) and then use it to derive the Lorentz transformations.

4.4 4-Velocity, 4-Acceleration

We now derive an equation, Eq. (4.36), for the "relativity acceleration," that is, the acceleration of a single particle parallel to $d\mathbf{s}$ using space-time algebra. We also obtain the relativistic acceleration perpendicular to the velocity. Four acceleration is a better word since it is simply the time derivative of the 4-velocity.

Four velocity $\mathbf{V}_4$ and four acceleration $\mathbf{A}_4$ will be denoted by $\mathbf{V}$ and $\mathbf{A}$ without a subscript.

4-Velocity is

$$\begin{aligned}
\mathbf{V} &= \gamma \frac{d\mathbf{s}}{dt} = \gamma \frac{d\left(ct\mathbf{e}_0 + x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3\right)}{dt} \qquad \gamma = \frac{1}{\left(1 - v^2/c^2\right)^{1/2}} \\
&= \gamma \left(\frac{cdt\mathbf{e}_0}{dt} + \frac{dx\mathbf{e}_1}{dt} + \frac{dy\mathbf{e}_2}{dt} + \frac{dz\mathbf{e}_3}{dt} \right) \\
\mathbf{V} &= \gamma \left(c\mathbf{e}_0 + v_x\mathbf{e}_1 + v_y\mathbf{e}_2 + v_z\mathbf{e}_3\right) = \gamma \left(c\mathbf{e}_0 + \mathbf{v}\right) \qquad (4.32) \\
\mathbf{V}^2 &= \frac{-c^2\left(1 - v^2/c^2\right)}{\left(1 - v^2/c^2\right)} = -c^2
\end{aligned}$$

Thus $\mathbf{V}^2$ is an absolute invariant.

4-momentum and 4-force are

$$\mathbf{P} = m\mathbf{V} \qquad \text{and} \qquad \mathbf{F} = d\mathbf{P}/dt$$

Differentiating $\mathbf{V}$ with respect to an invariant time, namely the time dt/γ measured in the moving frame gives the invariant 4-acceleration $\mathbf{A}$. The velocity measured in the moving frame is $\mathbf{v}$. Thus

4-Acceleration

$$\mathbf{A} = \gamma\frac{d\mathbf{V}}{dt} = \gamma\frac{d\left(\gamma c\mathbf{e}_0 + \gamma\mathbf{v}\right)}{dt} \tag{4.33}$$

$$\mathbf{A} = \gamma\left[\dot{\gamma}c\mathbf{e}_0 + \dot{\gamma}\mathbf{v} + \gamma\dot{\mathbf{v}}\right] = \gamma\left[\dot{\gamma}c\mathbf{e}_0 + \dot{\gamma}\mathbf{v} + \gamma\mathbf{a}\right] \tag{4.34}$$

$$\dot{\gamma} = \frac{d}{dt}\frac{1}{\left(1 - v^2/c^2\right)^{1/2}} = -\frac{2}{2}\frac{\left[-\left(\mathbf{v}\cdot\dot{\mathbf{v}}\right)\right]/c^2}{\left(1 - v^2/c^2\right)^{3/2}} = \frac{\gamma^3\left(\mathbf{v}\cdot\mathbf{a}\right)}{c^2} \tag{4.35}$$

$$\mathbf{A} = \left[\frac{\gamma^4\left(\mathbf{v}\cdot\mathbf{a}\right)\mathbf{e}_0}{c} + \frac{\gamma^4\left(\mathbf{v}\cdot\mathbf{a}\right)\mathbf{v}}{c^2} + \gamma^2\mathbf{a}\right] \tag{4.36}$$

$$\begin{aligned}
\mathbf{A}^2 &= \gamma^2\left[-\frac{\gamma^6\left(\mathbf{v}\cdot\mathbf{a}\right)^2 c^2}{c^4} + \frac{\gamma^6\left(\mathbf{v}\cdot\mathbf{a}\right)^2 v^2}{c^4} + \frac{2\gamma\gamma^3\left(\mathbf{v}\cdot\mathbf{a}\right)\left(\mathbf{v}\cdot\mathbf{a}\right)}{c^2} + \gamma^2\mathbf{a}^2\right] \\
&= \gamma^2\left[-\frac{\gamma^6\left(\mathbf{v}\cdot\mathbf{a}\right)^2\left(c^2 - v^2\right)}{c^4} + \frac{2\gamma\gamma^3\left(\mathbf{v}\cdot\mathbf{a}\right)\left(\mathbf{v}\cdot\mathbf{a}\right)}{c^2} + \gamma^2\mathbf{a}^2\right] \\
&= \gamma^2\left[-\frac{\gamma^6\left(\mathbf{v}\cdot\mathbf{a}\right)^2}{c^2}\left(1 - v^2/c^2\right) + \frac{2\gamma\gamma^3\left(\mathbf{v}\cdot\mathbf{a}\right)\left(\mathbf{v}\cdot\mathbf{a}\right)}{c^2} + \gamma^2\mathbf{a}^2\right] \\
&= \gamma^2\left[-\frac{\gamma^4\left(\mathbf{v}\cdot\mathbf{a}\right)^2}{c^2} + \frac{2\gamma^4\left(\mathbf{v}\cdot\mathbf{a}\right)^2}{c^2} + \gamma^2\mathbf{a}^2\right]
\end{aligned}$$

$$\mathbf{A}^2 = \gamma^4\left[\frac{\gamma^2\left(\mathbf{v}\cdot\mathbf{a}\right)^2}{c^2} + \mathbf{a}^2\right] \tag{4.37}$$

For $\mathbf{A}_4$ parallel to $\mathbf{v}$

$$\mathbf{A}_\parallel^2 = \gamma^4\left[\frac{v^2}{c^2\left(1 - v^2/c^2\right)} + 1\right]\mathbf{a}_\parallel^2 = \gamma^4\left[\frac{c^2 - v^2 + v^2}{c^2 - v^2}\right]\mathbf{a}_\parallel^2 = \gamma^4\frac{a_\parallel^2}{1 - v^2/c^2} = \gamma^6\mathbf{a}_\parallel^2$$

$$\mathbf{A}_\parallel = \gamma^3\mathbf{a}_\parallel = \frac{\mathbf{a}_\parallel}{\left(1 - v^2/c^2\right)^{3/2}} = \frac{d\mathbf{v}/dt}{\left(1 - v^2/c^2\right)^{3/2}} \tag{4.38}$$

The notation $\mathbf{A}_\parallel$ and $\mathbf{a}_\parallel$ denote the components of $\mathbf{A}$ and $\mathbf{a}$ parallel to $\mathbf{v}$. If $\mathbf{v}$ is parallel to the x-axis appearing in the Cartesian Lorentz transformations, then $\mathbf{v} = v_x \mathbf{e}_1$ in the direction of the velocity of separation w of coordinate axes S, S'.

The fact that $\mathbf{A}_\parallel^2$ is a scalar means that it is a Lorentz invariant (same value in all reference frames moving uniformly.) Rosser (1964) calls Eq. (4.38) the relativity acceleration and labels it $\boldsymbol{\phi}$. Thus

$$\mathbf{A}_\parallel = \boldsymbol{\phi} = \frac{d\mathbf{v}/dt}{(1 - v^2/c^2)^{3/2}} = \frac{d\mathbf{v}'/dt'}{(1 - v'^2/c^2)^{3/2}} \tag{4.39}$$

or

$$\mathbf{A}_\parallel = \boldsymbol{\phi} = \frac{\mathbf{a}_\parallel}{(1 - v^2/c^2)^{3/2}} = \frac{\mathbf{a}'_\parallel}{(1 - v'^2/c'^2)^{3/2}} \tag{4.40}$$

$\boldsymbol{\phi}$ is an invariant, that is, it has the same value relative to all reference systems having constant relative velocities in the same direction.

The components of $\mathbf{A}$ and $\mathbf{a}$ perpendicular to v are denoted by $\mathbf{A}_\perp$ and $\mathbf{a}_\perp$. The connection between them is:

$$\mathbf{A}_\perp = \gamma^2 \mathbf{a}_\perp \tag{4.41}$$

4.5 Orthogonality of A and V

Note that since the square of the velocity 4-vector is a constant, $-c^2$, we can form the derivative

$$\begin{aligned}
\frac{d\mathbf{V}^2}{d\tau} &= \frac{\gamma d\mathbf{V}^2}{dt} \qquad \gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \\
\frac{\gamma d\mathbf{V}}{dt} \cdot \mathbf{V} &= \mathbf{A} \cdot \mathbf{V} = 0 \\
\mathbf{V} &= \gamma \left(c\mathbf{e}_0 + \mathbf{v} \right) \\
\mathbf{A} &= \gamma \left(\dot{\gamma} c \mathbf{e}_0 + \dot{\gamma} \mathbf{v} + \gamma \mathbf{a} \right)
\end{aligned}$$

$$
\begin{aligned}
\mathbf{A}\cdot\mathbf{V} &= \left[-c^2\gamma^2\dot{\gamma}+\gamma^2\dot{\gamma}v^2+\gamma^3\left(\mathbf{a}\cdot\mathbf{v}\right)\right]=0\\
&= \gamma^2\left[-c^2\dot{\gamma}+\dot{\gamma}v^2+\gamma\left(\mathbf{a}\cdot\mathbf{v}\right)\right]=0\\
\dot{\gamma} &= \frac{\gamma^3\left(\mathbf{v}\cdot\mathbf{a}\right)}{c^2} \qquad 1-\frac{v^2}{c^2}=\frac{1}{\gamma^2}\\
\mathbf{A}\cdot\mathbf{V} &= \gamma^2\left[-c^2\dot{\gamma}\left(1-\frac{v^2}{c^2}\right)+\gamma\left(\mathbf{a}\cdot\mathbf{v}\right)\right]=0\\
&= \left[-c^2\frac{\gamma^3}{c^2}\left(\mathbf{v}\cdot\mathbf{a}\right)+\gamma^3\left(\mathbf{a}\cdot\mathbf{v}\right)\right]=0\\
&= \left(\mathbf{v}\cdot\mathbf{a}\right)\left[-\gamma^3+\gamma^3\right]=0
\end{aligned}
$$

Thus $\mathbf{A}\cdot\mathbf{V}=0$ does not say that $\mathbf{a}\cdot\mathbf{v}=0$,but it does say that

$$\mathbf{a}\cdot\mathbf{v}=av\cos\theta$$

$\mathbf{a}\cdot\mathbf{v}$ can have any value between av, for $\mathbf{a}$ and $\mathbf{v}$ parallel, and $\mathbf{a}\cdot\mathbf{v}=0$, when $\mathbf{a}$ and $\mathbf{v}$ are orthogonal. $\mathbf{A}\cdot\mathbf{V}=0$ is an invariant, namely zero.

4.6 Evaluation of A and A^2 when the Path is Curved

First consider non-relativistic velocity along a curve. Let $\mathbf{t}$ = a unit tangent to the curve.

$$\mathbf{v}=v\mathbf{t}$$

$$\frac{d\mathbf{v}}{dt}=\frac{d\left(v\mathbf{t}\right)}{dt}=\dot{v}\mathbf{t}+v\frac{d\mathbf{t}}{dt}=\dot{v}\mathbf{t}+v\frac{d\mathbf{t}}{d\ell}\frac{d\ell}{dt}$$

$$
\begin{aligned}
\mathbf{t}_1 &= \mathbf{t} \qquad \mathbf{t}_2=\mathbf{t}_1+d\mathbf{t}_1\\
d\ell &= rd\theta\\
d\mathbf{t} &= td\theta\hat{\mathbf{n}} \qquad t=1 \qquad \hat{\mathbf{n}}=\frac{\hat{\mathbf{r}}}{r}\\
\frac{d\mathbf{t}}{d\ell} &= \frac{d\theta}{d\ell}\hat{\mathbf{n}} \qquad rd\theta=d\ell \qquad \frac{d\theta}{d\ell}=\frac{1}{r}=\kappa=\text{curvature}\\
\frac{d\mathbf{t}}{d\ell} &= \frac{\hat{\mathbf{n}}}{r} \qquad \frac{d\ell}{dt}=v
\end{aligned}
$$

Then

$$
\begin{aligned}
\frac{d\mathbf{v}}{dt} &= \mathbf{a}=\dot{v}\mathbf{t}+v\frac{\hat{\mathbf{n}}}{r}v\\
\mathbf{a} &= \dot{v}\mathbf{t}+v^2\kappa\mathbf{n}
\end{aligned}
$$

$\ell =$ arc length

$$\frac{d\mathbf{t}}{d\ell} = \hat{\mathbf{n}}\frac{d\theta}{d\ell}, \qquad d\theta = \frac{d\ell}{r}, \qquad \frac{d\theta}{d\ell} = \frac{1}{r} = \kappa$$

$\mathbf{n} =$ unit vector along the normal to the path

$\kappa = \frac{1}{r} =$ curvature $\qquad \frac{d\ell}{dt} = v$

Thus

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = a_{\|}\mathbf{t} + v^2\kappa\mathbf{n} = a_{\|}\mathbf{t} + \frac{v^2}{r}\mathbf{n} = \mathbf{a}_{\|} + \mathbf{a}_{\perp} \tag{4.42}$$

Therefore
$$\mathbf{a}^2 = \left(\frac{d\mathbf{v}}{dt}\right)^2 = \dot{v}^2 + v^4\kappa^2$$

Therefore, going back to Eq. (4.37)
$$\begin{aligned}
\mathbf{A}^2 &= \frac{\gamma^6(\mathbf{v}\cdot\mathbf{a})^2}{c^2} + \gamma^4\mathbf{a}^2 \qquad (4.1)\\
&= \gamma^6\left[\frac{v\mathbf{t}\cdot(\dot{v}\mathbf{t} + v^2\kappa\mathbf{n})}{c^2}\right]^2 + \gamma^4\left(\dot{v}^2 + v^4\kappa^2\right)\\
&= \frac{\gamma^6(v\dot{v})^2}{c^2} + \gamma^4\dot{v}^2 + \gamma^4v^2\kappa^2\\
&= \gamma^6\dot{v}^2\left(\frac{v^2}{c^2} + \frac{1}{\gamma^2}\right) + \gamma^4v^2\kappa^2\\
\mathbf{A}^2 &= \gamma^6\dot{v}^2 + \gamma^4v^2\kappa^2
\end{aligned}$$

$$\mathbf{A}^2 = \gamma^4\left(\gamma^2\dot{v}^2 + v^2\kappa^2\right) \tag{4.44}$$

This is an "invariant," since $\mathbf{A}^2$ is invariant. $\mathbf{A} = d\mathbf{V}/d\tau = \frac{1}{(1-v^2/c^2)^{1/2}}\frac{dV}{dt}$.

$$\begin{aligned}
\mathbf{A} &= \gamma\left[\dot{\gamma}c\mathbf{e}_0 + \dot{\gamma}\mathbf{v} + \gamma\mathbf{a}\right]\\
\dot{\gamma} &= \frac{d}{dt}\frac{1}{(1-v^2/c^2)^{1/2}} = \frac{\mathbf{v}\cdot\frac{d\mathbf{v}}{dt}}{c^2(1-v^2/c^2)^{3/2}}\\
\frac{d\mathbf{v}}{dt} &= \dot{v}\mathbf{t} + v^2\kappa\mathbf{n}\\
\mathbf{v}\cdot\frac{d\mathbf{v}}{dt} &= v\mathbf{t}\cdot\left(\dot{v}\mathbf{t} + v^2\kappa\mathbf{n}\right) = v\dot{v} = va
\end{aligned}$$

Therefore, as in Eq. (4.36), only somewhat more general.

$$\dot{\gamma} = \frac{va}{c^2\left(1 - v^2/c^2\right)^{3/2}} = \frac{va\gamma^3}{c^2}$$
$$\mathbf{v} = v\mathbf{t}$$
$$\mathbf{a} = a\mathbf{t}+v^2\kappa\mathbf{n}$$

$$\mathbf{A} = \gamma\left[\frac{va\gamma^3\mathbf{e}_0}{c} + \frac{va\gamma^3\mathbf{v}}{c^2} + \gamma\mathbf{a}\right] \tag{4.45}$$

ℓ = arc length; $\mathbf{n}$ = unit vector along the normal to the path; $\kappa = \frac{1}{r}$ = curvature $\frac{d\ell}{dt} = v$

$$\frac{d\mathbf{t}}{d\ell} = \hat{\mathbf{n}}\frac{d\theta}{d\ell}, \qquad d\theta = \frac{d\ell}{r}, \qquad \frac{d\theta}{d\ell} = \frac{1}{r} = \kappa$$

Thus

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \dot{v}\mathbf{t} + v^2\kappa\mathbf{n} \tag{4.46}$$
$$\mathbf{a}^2 = \left(\frac{d\mathbf{v}}{dt}\right)^2 = \dot{v}^2 + v^4\kappa^2$$

$$\mathbf{A}^2 = \frac{\gamma^6\left(\mathbf{v}\cdot\mathbf{a}\right)^2}{c^2} + \gamma^4\mathbf{a}^2 = \gamma^6\left[\frac{v\mathbf{t}\cdot\left(\dot{v}\mathbf{t} + v^2\kappa\mathbf{n}\right)}{c^2}\right]^2 + \gamma^4\left(\dot{v}^2 + v^4\kappa^2\right)$$
$$= \frac{\gamma^6\left(v\dot{v}\right)^2}{c^2} + \gamma^4\dot{v}^2 + \gamma^4v^2\kappa^2 = \gamma^6\dot{v}^2\left(\frac{v^2}{c^2} + \frac{1}{\gamma^2}\right) + \gamma^4v^2\kappa^2$$
$$A^2 = \gamma^6\dot{v}^2 + \gamma^4v^2\kappa^2$$

Thus we have the relativistic result:

$$A^2 = \gamma^4\left(\dot{v}^2 + v^2\kappa^2\right) \tag{4.47}$$

This is a new "invariant," since A^2 is invariant. $\mathbf{A} = \gamma d\mathbf{V}/dt$, both $\mathbf{V}$ and dt/γ are invariants.

$$\mathbf{A}_{\parallel} = \gamma^3\mathbf{a}_{\parallel} \tag{4.49}$$
$$\mathbf{A}_{\perp} = \gamma^2\mathbf{a}_{\perp} = \gamma^2v^2\kappa\mathbf{n} \qquad \kappa = 1/\tau \tag{4.52}$$

The old, Galilean definition was

$$\mathbf{a} = \frac{d\mathbf{v}}{dt}$$

The above is not relativistically invariant. The term $\gamma^2\mathbf{a} = \mathbf{a}/\left(1 - v^2/c^2\right)$ appears as part of the coordinate acceleration terms in the 4-acceleration, $\mathbf{A}$, which is

$$\mathbf{A} = \mathbf{e}_0\gamma^4\left(\frac{\mathbf{v}\cdot\mathbf{a}}{c}\right) + \gamma^2\mathbf{a} + \frac{\gamma^4\left(\mathbf{v}\cdot\mathbf{a}\right)\mathbf{v}}{c^2}$$

The following is a simple independent derivation of Four Force.

$$\begin{aligned}
\mathbf{F}_4 &= m\frac{d}{d\tau}\mathbf{V} = m\frac{d}{d\tau}\gamma_v\left(\mathbf{e}_0 c + \mathbf{v}\right) \qquad \gamma_v = \frac{1}{\left(1 - v^2/c^2\right)^{1/2}} \\
\mathbf{F}_4 &= m\frac{d}{d\tau}\gamma_v\left(\mathbf{e}_0 c + \mathbf{v}\right) \qquad d\tau = \left(1 - v^2/c^2\right)^{1/2} dt \\
& \quad v = \mathbf{e}_1 v_x + \mathbf{e}_2 v_y + \mathbf{e}_3 v_z \qquad \frac{d}{d\tau} = \frac{1}{\left(1 - v^2/c^2\right)^{1/2}}\frac{d}{dt} \\
&= mA_4 = m\frac{1}{\left(1 - v^2/c^2\right)^{1/2}}\frac{d}{dt}\frac{1}{\left(1 - v^2/c^2\right)^{1/2}}\left(\mathbf{e}_0 c + \mathbf{v}\right) \\
&= m\gamma_v\left[-\frac{1}{2\left(1 - v^2/c^2\right)^{3/2}}\left(-\frac{2v}{c^2}\frac{dv}{dt}\right)\left(\mathbf{e}_0 c + \mathbf{v}\right) + \frac{\mathbf{a}}{\left(1 - v^2/c^2\right)^{1/2}}\right] \\
\mathbf{F}_4 &= m\gamma_v\left[\frac{\mathbf{e}_0\left(\mathbf{v}\cdot\mathbf{a}\right)}{c\left(1 - v^2/c^2\right)^{3/2}} + \frac{\left(\mathbf{v}\cdot\mathbf{a}\right)\mathbf{v}}{c^2\left(1 - v^2/c^2\right)^{3/2}} + \frac{\mathbf{a}}{\left(1 - v^2/c^2\right)^{1/2}}\right] \\
\mathbf{F}_4 &= m\left[\mathbf{e}_0\left(\frac{\mathbf{v}\cdot\mathbf{a}}{c}\right)\frac{1}{\left(1 - v^2/c^2\right)^2} + \frac{\left(\mathbf{v}\cdot\mathbf{a}\right)\mathbf{v}}{c^2\left(1 - v^2/c^2\right)^2} + \frac{\mathbf{a}}{\left(1 - v^2/c^2\right)}\right] \\
&= m\left[\mathbf{e}_0\frac{\left(\mathbf{v}\cdot\mathbf{a}\right)\gamma^4}{c} + \left(\mathbf{v}\cdot\mathbf{a}\right)\frac{\mathbf{v}}{c^2}\gamma^4 + \gamma^2\mathbf{a}\right]
\end{aligned}$$

4.7 Equation of Motion

To obtain the equation of motion of a body, equate the forces acting on it to the four-acceleration of the body (test particle) times its mass. In general the forces may be quite complicated, depending on the velocity and acceleration of both the test particle and depending also on the acceleration and velocity of the source body or even on a combination of velocity and acceleration of both bodies as in the Poynting vector in electromagnetism. In general the acceleration will not be in the same direction as the force.

When the force is parallel to the velocity, it is equated to the component of the 4-acceleratoin that is parallel to the velocity. This component is called the relativity acceleration $\boldsymbol{\phi}$ and is given by

$$\mathbf{a}_{\|} = \boldsymbol{\phi} = \gamma^3 \mathbf{a} \qquad \gamma = 1/\sqrt{1 - v^2/c^2}$$

$\boldsymbol{\phi}$ is an invariant and one may write

$$\mathbf{a}_{\|} = \boldsymbol{\phi} = \gamma^3 \mathbf{a} = \mathbf{a}'_{\|} = \gamma'^3 \mathbf{a}' \qquad \gamma' = 1/\sqrt{1 - v'^2/c^2}$$

where the prime denotes values in any other inertial frame. To obtain the equation of motion, in this case, the force or forces that are parallel to the acceleration and velocity $(\mathbf{f}_{\|})$ are equated to $m\boldsymbol{\phi}$. Thus

$$m\boldsymbol{\phi} = \mathbf{f}_{\|}$$

If the acceleration takes place in a constant gravitational field, then

$$m\phi = mg$$

where g is in the direction of the field; in this case it is the usual acceleration due to gravity.

4.8 Velocity as a Function of t

As shown by Rosser (1964), if initial velocity is in the direction of g or an electric field E and along the x-axis we have

$$\begin{aligned} \frac{d}{dt}\left(\frac{v_x}{\sqrt{1 - v_x^2/c^2}}\right) &= \frac{g}{m} \text{ (or } \frac{qE}{m}, \quad E = \text{electric field)} \\ \frac{v_x}{\sqrt{1 - v_x^2/c^2}} &= \frac{gt}{m} + C_1 \quad \text{or} \quad \frac{v_x}{\sqrt{1 - v_x^2/c^2}} = \frac{qEt + C_1}{m} \end{aligned}$$

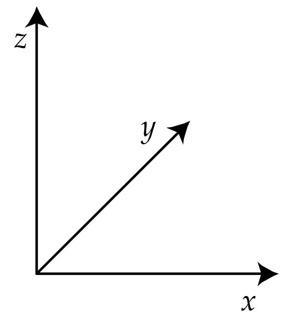

Fig. 4.3

If $v_x = 0$ at $t = 0$, then $C_1 = 0$ and

$$\begin{aligned}
v_x^2 &= \left(1 - \frac{v_x^2}{c^2}\right) \frac{q^2E^2t^2}{m^2} \\
\text{or for gravity} \quad v_x^3 &= v_x^2 = \left(1 - \frac{v_x^2}{c^2}\right) \frac{g^2t^2}{m^2} \\
\text{Thus} \qquad v_x^2 &= \left(1 - \frac{v_x^2}{c^2}\right) \ell^2 \qquad \ell^2 = \frac{q^2E^2t^2}{m^2} \quad \text{or} \quad \ell^2 = \frac{g^2t^2}{m^2} \\
v_x^2\left(1 + \frac{\ell^2}{c^2}\right) &= \ell^2 \qquad v_x^2 = \frac{\ell^2}{1 + \ell^2/c^2} \\
v_x^2 &= \frac{q^2E^2t^2/m^2}{1 + q^2E^2t^2/m^2c^2} \qquad qE \to mg \\
& \text{can replace } qE \text{ by } m\phi \\
v_x = \frac{dx}{dt} &= \frac{qEt}{m\left[1 + \dfrac{q^2E^2t^2}{m^2c^2}\right]^{1/2}} \text{ or for gravity } v_x = \frac{gt}{\left[1 + \dfrac{g^2t^2}{c^2}\right]^{1/2}} \qquad (4.50)
\end{aligned}$$

For distance traveled

$$x = \int_0^t \frac{qEtdt}{m\left[1 + q^2E^2t^2/mc^2\right]^{1/2}} = \frac{mc^2}{qE}\left[1 + \frac{q^2E^2t^2}{m^2c^2}\right]^{1/2} + C_2$$

If $x = 0$ at $t = 0$

$$O = \frac{mc^2}{qE} + C_2 \qquad C_2 = -\frac{mc^2}{qE}$$

Electric field

$$x = \frac{mc^2}{qE}\left[\left(1 + \frac{q^2E^2t^2}{mc^2}\right) - 1\right] \qquad (4.51)$$

Gravity

$$x = \frac{c^2}{g}\left[\left(1 + \frac{g^2t^2}{c^2}\right) - 1\right] \tag{4.52}$$

By Eq. (4.50), if $gt \ll c$

$$v_x = gt$$

Maximum velocity.

In

$$v_x^2 = \underset{t\to\infty}{L}\left(\frac{g^2t^2}{1+\dfrac{g^2t^2}{c^2}}\right) \tag{4.53}$$

Differentiate numerator and denominator twice with respect to t.

$$\begin{aligned} v_x^2 &= \frac{2g^2}{2\dfrac{g^2}{c^2}} = c^2 \\ v_x &= c \end{aligned} \tag{4.54}$$

In Eq. (4.50) or Eq. (4.51), if $\dfrac{qEt}{m}$ or $gt \ll c$

$$\begin{aligned} x &= \frac{mc^2}{qE}\left[1 + \frac{q^2E^2t^2}{2mc^2} + ...\right] \\ x &= \frac{1}{2}\left(\frac{qE}{m}\right)t^2 \quad \text{or} \quad x = \frac{1}{2}gt^2 \end{aligned} \tag{4.55}$$

4.9 Total Work Done on Charge by Electric Field or Gravitational Field

$$\begin{aligned} \text{Work done} &= \int F_x dx = \int qE_x dx = \int mg dx \\ E_x &= -\frac{\partial V}{\partial x_x} \qquad V = \text{electrostatic potential} \\ \text{Work done on charge} &= -q\int_0^x \frac{\partial V}{\partial x}dx = -q\int_{V=V_0}^{V=0} dV = qV_0 \ \text{ joules} \end{aligned}$$

or

$$W = mV_0$$

Equating work done on charge to gain in KE

$$T = qV_0 \qquad \text{or} \qquad T = mV_0$$

For general case, where initial velocity $= v_0$,

$$T - T_0 = \frac{mc^2}{1 - v^2/c^2} - \frac{mc^2}{1 - v_0^2/c^2}$$

In the preceding equations replace T by $T - T_0$ and mc^2 by $\dfrac{mc^2}{\sqrt{1 - v_0^2/c^2}}$.

4.10 Equation of Motion of a Non-Rotating Mass Pulled Into a Black Hole (Flat Space)

Relativity acceleration is

$$\begin{aligned}
\phi &= \frac{a'}{(1 - v^2/c^2)^{3/2}} && a' = \text{acceleration at a given time.} \\
\phi dx &= \frac{a' dx}{(1 - v^2/c^2)^{3/2}} && dx = \frac{dx}{dt}\frac{dt}{dv}dv \qquad dv = \frac{v}{a}dv \\
\phi dx &= c^2 \frac{a}{a} \frac{\frac{xv}{c}\frac{dv}{c}}{(1 - v^2/c^2)^{3/2}} = \frac{c^2 y dy}{(1 - y^2)^{3/2}} && y = \frac{v}{c} \\
\int \phi dx &= c^2 \int \frac{y dy}{(1 - y^2)^{3/2}} = c^2 \left[\frac{1}{\sqrt{1 - y^2}} \right]_{y_0}^{y} \\
\phi &= \frac{GM}{x^2 (1 - a)^{1/2}} && a = \frac{GM}{c^2 x} \\
GM \int_x^{x_0} \frac{1}{x^2} &= c^2 \left[\frac{1}{\sqrt{1 - v^2/c^2}} - \frac{1}{\sqrt{1 - v_0^2/c^2}} \right] \\
\text{Let } v_0 &= 0 \quad \frac{1}{\sqrt{1 - v^2/c^2}} = 1 + \frac{GM}{c^2} \left(\frac{1}{R} - \frac{1}{R_0} \right)
\end{aligned}$$

$$\begin{aligned}
x_0 &= R_0 = \text{initial distance from black hole} \\
M &= \text{mass of black hole} \\
R &= \text{radius of black hole} \\
v &= \text{velocity at radius of black hole}
\end{aligned}$$

$$\begin{aligned}
\frac{dv}{dt} &= g = \frac{GM}{R^2} \\
\text{Let} \quad R_0 &= nR \qquad \frac{1}{R} - \frac{1}{nR} = \frac{(n-1)}{nR}
\end{aligned}$$

$$\begin{aligned}
\frac{1}{\sqrt{1 - v^2/c^2}} &= 1 + \frac{GM\,(n-1)}{R} \\
\frac{1}{\sqrt{1 - v^2/c^2}} - 1 &= \frac{GM\,(n-1)}{R}. \quad \text{If } v \ll c \quad \frac{1}{\sqrt{1 - v^2/c^2}} \simeq 1 + \frac{v^2}{2c^2} \\
\frac{v^2}{2c^2} &\simeq \frac{GM\,(n-1)}{R} \\
v &= c\left(\frac{2GM\,(n-1)}{R}\right)^{1/2} \\
\frac{v^4}{4c^4} &= \frac{G^2M^2\,(n-1)^2}{R^2} \\
\frac{GM}{R^2} &= \frac{v^4}{4c^4\,(n-1)^2\,GM} \\
\frac{dv}{dt} &= \frac{v^4}{4c^2\,(n-1)^2\,GM} \\
\frac{dv}{v^4} &= \frac{dt}{4c^2\,(n-1)^2\,GM} \\
\left[\frac{v^5}{5}\right]_{v_0}^{v} &= \frac{t - t_0}{4c^2\,(n-1)^2\,GM} \\
\frac{v^5 - v_0^5}{5} &= \frac{t - t_0}{4c^2\,(n-1)^2\,GM}
\end{aligned}$$

For $t_0 = 0, \quad v_0 = 0$

$$v^5 = \frac{5t}{4c^2\,(n-1)^2\,GM} \tag{4.56}$$

$$\begin{aligned}
\frac{dR}{dt} &= v \\
dR &= vdt \\
dR &= \left(\frac{5}{4c^2\,(n-1)^2\,GM}\right)^{1/5} t^{1/5}dt = Kt^{1/5}dt
\end{aligned}$$

$$\begin{aligned}
\int t^n dt &= \frac{t^{n+1}}{n+1} \\
R &= \frac{Kt^{\frac{1}{5}+1}}{\frac{1}{5}+1} = \frac{5Kt^{6/5}}{6} \\
K &= \left(\frac{5}{4c^2\left(n-1\right)^2 GM}\right)^{1/5}
\end{aligned} \tag{4.57}$$

For a discussion within the framework of general relativity of black holes and motion in a Schwarzschild metric, see Taylor and Wheeler (2000).

4.11 Transformation of Electromagnetic Bivector under Lorentz Transformation

For the usual procedure leading to these results, see Rosser (1964). In the following, the Lorentz transformation rotates the unit vectors that define the electromagnetic field bivector. All the $\mathbf{E}$ components are divided by c, which factor we temporarily omit. After rotation, the bivector has different coefficients but is still defined in terms of the same unit vectors. Let $L = \mathbf{e}^{\mathbf{e}_0\mathbf{e}_1\theta/2}$,
$L^{-1} = \mathbf{e}^{-\mathbf{e}_0\mathbf{e}_1\theta/2}$. Recall $e^{\mathbf{e}_0\mathbf{e}_1\theta} = \mathbf{e}_0 \cosh\theta + \mathbf{e}_1 \sinh\theta$ and
$\cosh^2\theta - \sinh^2\theta = 1$

$$\begin{aligned}
\mathbf{F}' &= E'_x\mathbf{e}_0\mathbf{e}_1 + E'_y\mathbf{e}_0\mathbf{e}_2 + E'_z\mathbf{e}_0\mathbf{e}_3 + B'_z\mathbf{e}_1\mathbf{e}_2 + B'_y\mathbf{e}_3\mathbf{e}_1 + B'_x\mathbf{e}_2\mathbf{e}_3 \\
\mathbf{F} &= E_x\mathbf{e}_0\mathbf{e}_1 + E_y\mathbf{e}_0\mathbf{e}_2 + E_z\mathbf{e}_0\mathbf{e}_3 + B_z\mathbf{e}_1\mathbf{e}_2 + B_y\mathbf{e}_3\mathbf{e}_1 + B_x\mathbf{e}_2\mathbf{e}_3 \\
\mathbf{F}' &= e^{-\mathbf{e}_0\mathbf{e}_1\theta/2}[E_x\mathbf{e}_0\mathbf{e}_1 + E_y\mathbf{e}_0\mathbf{e}_2 + E_z\mathbf{e}_0\mathbf{e}_3 \\
&\quad +B_z\mathbf{e}_1\mathbf{e}_2 + B_y\mathbf{e}_3\mathbf{e}_1 + B_x\mathbf{e}_2\mathbf{e}_3]e^{\mathbf{e}_0\mathbf{e}_1\theta/2} \\
&= L^{-1}\mathbf{F}L = E_xL^{-1}\mathbf{e}_0\mathbf{e}_1L + -- = E_xL^{-1}\mathbf{e}_0LL^{-1}\mathbf{e}_1L + ---\\
\mathbf{F}' &= \mathbf{e}^{-\mathbf{e}_0\mathbf{e}_1\theta/2}\mathbf{F}\mathbf{e}^{\mathbf{e}_0\mathbf{e}_1\theta/2}
\end{aligned}$$

Consider

$$\mathbf{e}^{-\mathbf{e}_0\mathbf{e}_1\theta/2}\mathbf{e}_0\mathbf{e}^{\mathbf{e}_0\mathbf{e}_1\theta/2} = \mathbf{e}^{-\mathbf{e}_0\mathbf{e}_1\theta}\mathbf{e}_0, \qquad \mathbf{e}^{-\mathbf{e}_0\mathbf{e}_1\theta/2}\mathbf{e}_1\mathbf{e}^{\mathbf{e}_0\mathbf{e}_1\theta/2} = \mathbf{e}^{-\mathbf{e}_0\mathbf{e}_1\theta}\mathbf{e}_1, \text{ etc.}$$

Then

$$\begin{aligned}
\mathbf{F}' &= E_x\left(e^{-\mathbf{e}_0\mathbf{e}_1\theta}\mathbf{e}_0\right)\left(e^{-\mathbf{e}_0\mathbf{e}_1\theta}\mathbf{e}_1 + ---\right) \\
e^{-\mathbf{e}_0\mathbf{e}_1\theta}\mathbf{e}_0 &= \left(\cosh\theta - \mathbf{e}_0\mathbf{e}_1\sinh\theta\right)\mathbf{e}_0 = \mathbf{e}_0\cosh\theta - \mathbf{e}_1\sinh\theta \\
e^{-\mathbf{e}_0\mathbf{e}_1\theta}\mathbf{e}_1 &= \left(\cosh\theta - \mathbf{e}_0\mathbf{e}_1\sinh\theta\right)\mathbf{e}_1 = \mathbf{e}_1\cosh\theta - \mathbf{e}_0\sinh\theta \\
e^{\mathbf{e}_0\mathbf{e}_1\theta}\mathbf{e}_2 &= \left(\cosh\theta - \mathbf{e}_0\mathbf{e}_1\sinh\theta\right)\mathbf{e}_2 = \mathbf{e}_2\cosh - \mathbf{e}_0\mathbf{e}_1\mathbf{e}_2\sinh\theta
\end{aligned}$$

$$
\begin{aligned}
L^{-1}E_x\mathbf{e}_0\mathbf{e}_1 L &= E_x\left(\mathbf{e}_0\cosh\theta-\mathbf{e}_1\sinh\theta\right)\left(-\mathbf{e}_0\sinh\theta+\mathbf{e}_1\cosh\theta\right)\\
&= E_x\left(\sinh\theta\cosh\theta+\mathbf{e}_0\mathbf{e}_1\cosh^2\theta-\mathbf{e}_0\mathbf{e}_1\sinh^2\theta-\sinh\theta\cosh\theta\right)\\
&= E_x\left(\cosh^2\theta-\sinh^2\theta\right)\mathbf{e}_0\mathbf{e}_1 = E_x\mathbf{e}_0\mathbf{e}_1\\
e^{-\mathbf{e}_0\mathbf{e}_1\theta/2}\mathbf{e}_2 e^{\mathbf{e}_0\mathbf{e}_1\theta/2} &= e^{-\mathbf{e}_0\mathbf{e}_1\theta/2}\mathbf{e}_2\left(\cosh\theta/2+\mathbf{e}_0\mathbf{e}_1\sinh\theta/2\right)\\
&= \left(\cosh\theta/2-\mathbf{e}_0\mathbf{e}_1\sinh\theta/2\right)\left(\mathbf{e}_2\cosh\theta/2+\mathbf{e}_0\mathbf{e}_1\mathbf{e}_2\sinh\theta/2\right)\\
&= \mathbf{e}_2\cosh^2\theta/2+\mathbf{e}_0\mathbf{e}_1\mathbf{e}_2\sinh\theta/2\cosh\theta/2\\
&\quad -\mathbf{e}_0\mathbf{e}_1\mathbf{e}_2\sinh\theta/2\cosh\theta/2+\mathbf{e}_2\sinh\theta/2\sin\theta/2\\
&= \mathbf{e}_2\left(\cosh^2\theta/2-\sinh^2\theta/2\right)+0=\mathbf{e}_2
\end{aligned}
$$

Thus

$$
\begin{aligned}
E_xL^{-1}\mathbf{e}_0\mathbf{e}_1L &= E_x\mathbf{e}_0\mathbf{e}_1\\
E_yL^{-1}\mathbf{e}_0\mathbf{e}_2L &= E_yL^{-1}\mathbf{e}_0L\;L^{-1}\mathbf{e}_2L\\
&= E_y\left(\mathbf{e}_0\cosh\theta-\mathbf{e}_1\sinh\theta\right)\mathbf{e}_2\\
&= E_y\left(\mathbf{e}_0\mathbf{e}_2\cosh\theta-\mathbf{e}_1\mathbf{e}_2\sinh\theta\right)\\
E_zL^{-1}\mathbf{e}_0\mathbf{e}_3L &= E_z\left(L^{-1}\mathbf{e}_0L\right)\left(L^{-1}\mathbf{e}_3L\right)\\
&= E_z\mathbf{e}_0\mathbf{e}_3\cosh\theta-\mathbf{e}_1\mathbf{e}_3\sinh\theta = E_z\left(\mathbf{e}_0\mathbf{e}_3\cosh\theta+\mathbf{e}_3\mathbf{e}_1\sinh\theta\right)\\
B_zL^{-1}\mathbf{e}_1\mathbf{e}_2L &= B_zL^{-1}\mathbf{e}_1L\;L^{-1}\mathbf{e}_2L\\
&= B_z\left(\mathbf{e}_1\cosh\theta-\mathbf{e}_0\sinh\theta\right)\mathbf{e}_2 = B_z\cosh\theta\mathbf{e}_1\mathbf{e}_2-B_z\sinh\theta\mathbf{e}_0\mathbf{e}_2\\
B_yL^{-1}\mathbf{e}_3\mathbf{e}_1L &= B_yL^{-1}\mathbf{e}_3L\;L^{-1}\mathbf{e}_1L\\
&= B_y\mathbf{e}_3\left(\mathbf{e}_1\cosh\theta-\mathbf{e}_0\sinh\theta\right)\\
&= B_y\mathbf{e}_3\mathbf{e}_1\cosh\theta+B_y\mathbf{e}_0\mathbf{e}_3\sinh\theta\\
B_xL^{-1}\mathbf{e}_2\mathbf{e}_3L &= B_xL^{-1}\mathbf{e}_2L\;L^{-1}\mathbf{e}_3L = B_x\mathbf{e}_2\mathbf{e}_3
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{F}' &= E'_x\mathbf{e}_0\mathbf{e}_1+E'_y\mathbf{e}_0\mathbf{e}_2+E'_z\mathbf{e}_0\mathbf{e}_3+B'_z\mathbf{e}_1\mathbf{e}_2+B'_y\mathbf{e}_3\mathbf{e}_1+B'_x\mathbf{e}_2\mathbf{e}_3\\
&= L^{-1}\mathbf{F}L = E_x\mathbf{e}_0\mathbf{e}_1+E_y\cosh\theta\mathbf{e}_0\mathbf{e}_2-E_y\sinh\theta\mathbf{e}_1\mathbf{e}_2\\
&\quad +E_z\cosh\theta\mathbf{e}_0\mathbf{e}_3+E_z\sinh\theta\mathbf{e}_3\mathbf{e}_1\\
&\quad +B_z\cosh\theta\mathbf{e}_1\mathbf{e}_2-B_z\sinh\theta\mathbf{e}_0\mathbf{e}_2\\
&\quad +B_y\cosh\theta\mathbf{e}_3\mathbf{e}_1-B_y\sinh\theta\mathbf{e}_0\mathbf{e}_3+B_x\mathbf{e}_2\mathbf{e}_3\\
&= E_x\mathbf{e}_0\mathbf{e}_1+\left(E_y\cosh\theta-B_z\sinh\theta\right)\mathbf{e}_0\mathbf{e}_2\\
&\quad +\left(E_z\cosh\theta+B_y\sinh\theta\right)\mathbf{e}_0\mathbf{e}_3+\left(-E_y\sinh\theta+B_z\cosh\theta\right)\mathbf{e}_1\mathbf{e}_2\\
&\quad +\left(E_z\sinh\theta+B_y\cosh\theta\right)\mathbf{e}_3\mathbf{e}_1+B_x\mathbf{e}_2\mathbf{e}_3
\end{aligned}
$$

If we use $\cosh\theta = \gamma, \quad \sinh = \gamma\beta, \quad \tanh\theta = \beta, \quad \beta = \mathtt{w}/c$ where the velocity of separation S' relative to S is denoted by $\mathtt{w}$,then equating coefficients of the unit vector products $\mathbf{e}_0\mathbf{e}_1$, $\mathbf{e}_0\mathbf{e}_2$ etc. and dividing the E's by c yields:

$$
\begin{aligned}
\frac{E'_x}{c} &= \frac{E_x}{c} \\
\frac{E'_y}{c} &= \frac{E_y}{c}\cosh\theta - B_z\sinh\theta = \gamma\left(\frac{E_y}{c} - \beta B_z\right) = \gamma\left(\frac{E_y}{c} - \frac{\mathtt{w}}{c}B_z\right) \\
\frac{E'_z}{c} &= \frac{E_z}{c}\cosh\theta + B_y\sinh\theta = \gamma\left(\frac{E_z}{c} + \beta B_y\right) = \gamma\left(\frac{E_z}{c} + \frac{\mathtt{w}}{c}B_y\right) \\
B'_z &= B_z\cosh\theta - \frac{E_y}{c}\sinh\theta = \gamma\left(B_z - \frac{E_y}{c}\beta\right) = \gamma\left(B_z - \frac{\mathtt{w}}{c}\frac{E_y}{c}\right) \\
B'_y &= B_y\cosh\theta + \frac{E_z}{c}\sinh\theta = \gamma\left(B_y - \frac{E_z}{c}\beta\right) = \gamma\left(B_y - \frac{\mathtt{w}}{c}\frac{E_z}{c}\right) \\
B'_x &= B_x
\end{aligned}
$$

Thus **E** and **B** do not transform like vectors or 4-vectors, but transform like tensors. Changing the sign of $\mathtt{w}$ gives the field transformations from S' to S.

4.12 Summary

The transformations and their converses are:

$$
\begin{array}{ll}
E'_x = E_x & E_x = E'_x \\
E'_y = \gamma\left(E_y - \mathtt{w}B_z\right) & E_y = \gamma\left(E'_y + \mathtt{w}B'_z\right) \\
E'_z = \gamma\left(E_z + \mathtt{w}B_y\right) & E_z = \gamma\left(E'_z - \mathtt{w}B'_y\right)
\end{array}
$$

$$
\begin{array}{ll}
B'_x = B_x & B_x = B'_x \\
B'_y = \gamma\left(B_y + \frac{\mathtt{w}}{c^2}E_z\right) & B_y = \gamma\left(B'_y - \frac{\mathtt{w}}{c^2}E'_z\right) \\
B'_z = \gamma\left(B_z - \frac{\mathtt{w}}{c^2}E_y\right) & B_z = \gamma\left(B'_z + \frac{\mathtt{w}}{c^2}E'_y\right)
\end{array}
$$

In vector form the above may be written

$$
\begin{array}{ll}
\mathbf{E}'_{\parallel} = \left(\mathbf{E} + \mathbf{w}\times\mathbf{B}\right)_{\parallel} & \mathbf{E}'_{\perp} = \gamma\left(\mathbf{E}' + \mathbf{w}\times\mathbf{B}\right)_{\perp} \\
\mathbf{B}'_{\parallel} = \left(\mathbf{B} - \mathbf{w}\times\mathbf{E}/c^2\right)_{\parallel} & \mathbf{B}'_{\perp} = \gamma\left(\mathbf{B} - \mathbf{w}\times\mathbf{E}/c^2\right)_{\perp}
\end{array}
$$

The subscripts $\|$and $\perp$ represent components parallel and perpendicular to $\mathbf{w}$. $\gamma = \sqrt{1 - \mathtt{w}^2/c^2}$.