3

DEFINITION OF ASSOCIATIVE OR DIRECT PRODUCT AND ROTATION OF VECTORS

This chapter summarizes a few properties of Clifford Algebra and describe its usefulness in effecting vector rotations.

3.1 Definition of Associative or Direct Product

Given two vectors **a** and **b**, the resultant is

$$\mathbf{a} + \mathbf{b} = \mathbf{c}$$

We define the associative or direct product

$$\mathbf{cc} = (\mathbf{a} + \mathbf{b})(\mathbf{a} + \mathbf{b}) = \mathbf{aa} + \mathbf{bb} + \mathbf{ab} + \mathbf{ba}$$

For the special case **b** perpendicular to **a**, for which we write $\mathbf{b}_\perp$, Fig. 3.1b, this may be written

$$\mathbf{cc} = \mathbf{aa} + \mathbf{b}_\perp\mathbf{b}_\perp + \mathbf{ab}_\perp + \mathbf{b}_\perp\mathbf{a}$$

By definition, the direct product of a vector by itself is its scalar value squared. Thus, $\mathbf{cc} = c^2$, $\mathbf{b}_\perp\mathbf{b}_\perp = b_\perp^2$, . From the Pythagorean theorem in Euclidean flat space, $c^2 = a^2 + b_\perp^2$. Therefore we must have

$$\mathbf{ab}_\perp + \mathbf{b}_\perp\mathbf{a} = 0 \tag{3.1}$$

which says that perpendicular vectors anticommute.

In general, $\mathbf{b}$ can be expressed as the vector sum of $\mathbf{b}_{\|}$ and $\mathbf{b}_{\perp}$, parallel and perpendicular, respectively, to $\mathbf{a}$. $\mathbf{b}_{\|}$ is simply some scalar multiplier m times $\mathbf{a}$. Therefore,

$$\begin{aligned} \mathbf{a}\mathbf{b}_{\|} &= \mathbf{a}m\mathbf{a} = m\mathbf{a}\mathbf{a} \\ \mathbf{a}\mathbf{b}_{\|} &= \mathbf{b}_{\|}\mathbf{a} \end{aligned} \tag{3.2}$$

Eqs. (3.1) and (3.2) now define the rules of a geometric vector algebra in 2-dimensions.

3.2 Law of Cosines

Now apply the properties defined by Eqs. (3.1) and (3.2) to deduce the law of cosines. Referring to Fig. 3.1c,

$$\mathbf{c} = \mathbf{a} + \mathbf{b}$$

from which

$$\mathbf{c}\mathbf{c} = \mathbf{a}\mathbf{a} + \mathbf{b}\mathbf{b} + \mathbf{a}\mathbf{b} + \mathbf{b}\mathbf{a}$$

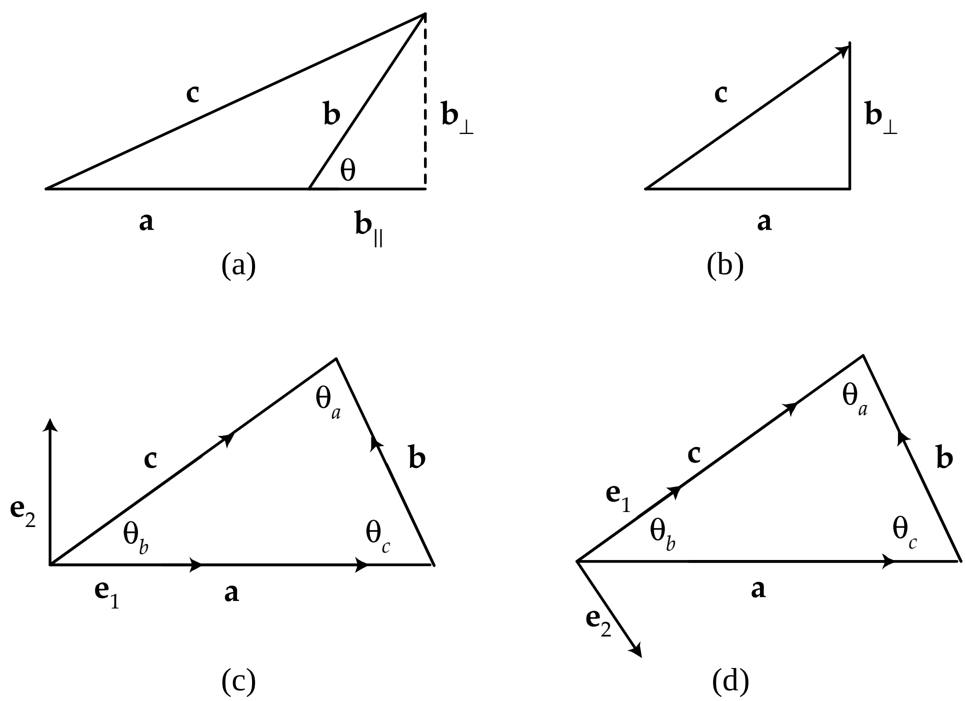

Fig. 3.1

Decomposing **b** in the last two terms into its components parallel and perpendicular to **a**

$$\mathbf{cc} = \mathbf{aa} + \mathbf{bb} + \mathbf{ab}_{\parallel} + \mathbf{ab}_{\perp} + \mathbf{b}_{\parallel}\mathbf{a} + \mathbf{b}_{\perp}\mathbf{a}$$

Using Eqs. (3.1) and (3.2)

$$c^2 = a^2 + b^2 + 2ab_{\parallel} \qquad\qquad c^2 = a^2 + b^2 + 2ab\cos\theta$$

where the quantities are now scalar values.

3.3 Law of Sines

$$\begin{aligned} \mathbf{b}_{\perp} &= \mathbf{e}_2 b \sin\theta_c \\ \mathbf{c}_{\perp} &= \mathbf{e}_2 c \sin\theta_b \\ \mathbf{b}_{\perp} &= \mathbf{c}_{\perp} \rightarrow \quad \mathbf{e}_2 b \sin\theta_c = \mathbf{e}_2 c \sin\theta_b \rightarrow \quad b\sin\theta_c = c\sin\theta_b \end{aligned}$$

Thus

$$\frac{b}{c} = \frac{\sin\theta_b}{\sin\theta_c}$$

and therefore

$$\frac{a}{c} = \frac{\sin\theta_a}{\sin\theta_c}, \quad \text{and} \quad \frac{a}{b} = \frac{\sin\theta_a}{\sin\theta_b}$$

3.4 Clifford Algebra in 3-Dimensions

Consider a position vector $\mathbf{x}$ where $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ are unit vectors along 3 orthogonal axes.

$$\mathbf{x} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3$$

Form the direct product

$$\begin{aligned} \mathbf{xx} &= (v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3)(v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3) = v_1^2\mathbf{e}_1\mathbf{e}_1 + v_2^2\mathbf{e}_2\mathbf{e}_2 + v_3^2\mathbf{e}_3\mathbf{e}_3 \\ &\quad + v_1v_2(\mathbf{e}_1\mathbf{e}_2 + \mathbf{e}_2\mathbf{e}_1) + v_1v_3(\mathbf{e}_1\mathbf{e}_3 + \mathbf{e}_3\mathbf{e}_1) + v_2v_3(\mathbf{e}_2\mathbf{e}_3 + \mathbf{e}_3\mathbf{e}_2) \end{aligned}$$

Since

$$\mathbf{x}^2 = v_1^2 + v_2^2 + v_3^2$$

which is the Pythagorean theorem in 3-dimensions, the properties of the unit vectors must be

$$\mathbf{e}_1\mathbf{e}_1 = \mathbf{e}_2\mathbf{e}_2 = \mathbf{e}_3\mathbf{e}_3 = 1 \qquad \mathbf{e}_1\mathbf{e}_2 + \mathbf{e}_2\mathbf{e}_1 = 0, \text{ etc. That is, } \mathbf{e}_i\mathbf{e}_j = -\mathbf{e}_j\mathbf{e}_i$$

The unit vectors are called the generators of the geometric algebra and in this case we use Cartesian unit vectors. A convenient way to generate all of the independent combinations in the algebra is to form the product

$$(1+\mathbf{e}_1)(1+\mathbf{e}_2)(1+\mathbf{e}_3) = \underset{\text{scalar}}{1} + \mathbf{e}_1 + \underset{\text{vector}}{\mathbf{e}_2} + \mathbf{e}_3 + \underbrace{\mathbf{e}_1\mathbf{e}_2 + \mathbf{e}_2\mathbf{e}_3 + \mathbf{e}_3\mathbf{e}_1}_{\text{bivector}} + \underset{\text{pseudo-scalar}}{\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3}$$

The result is an 8 element algebra. The elements consist of a scalar, 3 unit vectors that define lines, 3 bivectors that define oriented planes and a trivector $\mathbf{e} \equiv \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$ that corresponds to a volume. It has the property $\mathbf{ee} = -1$ and is called a pseudo-scalar. It also changes sign under inversion since $(-\mathbf{e}_1)(-\mathbf{e}_2)(-\mathbf{e}_3) = -\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$.

Note that the unit bivectors anticommute. For example, $(\mathbf{e}_1\mathbf{e}_2)(\mathbf{e}_2\mathbf{e}_3) = (\mathbf{e}_2\mathbf{e}_1\mathbf{e}_3\mathbf{e}_2) = -(\mathbf{e}_2\mathbf{e}_3)(\mathbf{e}_1\mathbf{e}_2)$. In this sense, they behave like vectors; however, their squares are -1. For example, $\mathbf{e}_1\mathbf{e}_2\mathbf{e}_1\mathbf{e}_2 = -\mathbf{e}_1\mathbf{e}_1\mathbf{e}_2\mathbf{e}_2 = -1$.

When negative values for each of the generators are included, the 16 elements form a group. Define, $\mathbf{e} \equiv \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$.

Note that the associative product of $\mathbf{e}$ with a bivector is a vector. For example:

$$\mathbf{ee}_1\mathbf{e}_2 = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\mathbf{e}_1\mathbf{e}_2 = \mathbf{e}_2\mathbf{e}_3\mathbf{e}_2 = \mathbf{e}_3.$$

Therefore, a general bivector

$$\mathbf{B} = \mathbf{e}_1\mathbf{e}_2 B_{12} + \mathbf{e}_2\mathbf{e}_3 B_{23} + \mathbf{e}_3\mathbf{e}_1 B_{31}$$

may be written

$$\begin{aligned} \mathbf{B} &= \mathbf{e}\left[-\mathbf{e}\left(\mathbf{e}_1\mathbf{e}_2 B_{12} + \mathbf{e}_2\mathbf{e}_3 B_{23} + \mathbf{e}_3\mathbf{e}_1 B_{31}\right)\right] \\ \mathbf{B} &= \mathbf{e}\left(\mathbf{e}_3 B_{12} + \mathbf{e}_1 B_{23} + \mathbf{e}_2 B_{31}\right) \end{aligned}$$

Since $\mathbf{ee}_3 = \mathbf{e}_1\mathbf{e}_2$, $\mathbf{ee} = \mathbf{e}_2\mathbf{e}_3$, $\mathbf{ee}_2 = \mathbf{e}_3\mathbf{e}_1$. $\mathbf{e}^2 = -1$. Thus $\mathbf{B} = \mathbf{ev}$, where $\mathbf{v}$ is a vector.

A linear combination of the elements of the algebra with arbitrary scalar multipliers, real or complex, is called a multivector. Thus, a space-like multivector may be written

$$\mathbf{M} = S + \mathbf{v} + \mathbf{ev} + S'\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$$

We now form the direct product of 2 vectors and express the result in terms of the vector components: $\mathbf{uv} = (\mathbf{uv} + \mathbf{vu})/2 + (\mathbf{uv} - \mathbf{vu})/2$

Express the vectors $\mathbf{u}, \mathbf{v}$ in terms of $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$

$$
\begin{aligned}
\mathbf{vu} &= (v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3)(u_1\mathbf{e}_1 + u_2\mathbf{e}_2 + u_3\mathbf{e}_3) \\
&= v_1u_1 + v_1u_2\mathbf{e}_1\mathbf{e}_2 + v_2u_1\mathbf{e}_2\mathbf{e}_1 + v_2u_2 + v_1u_3\mathbf{e}_1\mathbf{e}_3 \\
&\quad + v_3u_1\mathbf{e}_3\mathbf{e}_1 + v_3u_3 + v_2u_3\mathbf{e}_2\mathbf{e}_3 + v_3u_2\mathbf{e}_3\mathbf{e}_2 \\
\mathbf{uv} &= v_1u_1 + u_2v_1\mathbf{e}_2\mathbf{e}_1 + u_1v_2\mathbf{e}_1\mathbf{e}_2 + v_2u_2 + u_3v_1\mathbf{e}_3\mathbf{e}_1 \\
&\quad + u_1v_3\mathbf{e}_1\mathbf{e}_3 + v_3u_3 + u_3v_2\mathbf{e}_3\mathbf{e}_2 + u_2v_3\mathbf{e}_2\mathbf{e}_3
\end{aligned}
$$

$$
\begin{aligned}
\text{Thus, } \frac{1}{2}(\mathbf{vu} + \mathbf{uv}) &= \mathbf{v} \bullet \mathbf{u} = v_1u_1 + v_2u_2 + v_3u_3 \\
\text{and } \frac{1}{2}(\mathbf{vu} - \mathbf{uv}) &= \mathbf{v} \wedge \mathbf{u} \\
&= \frac{1}{2}[v_1u_2(\mathbf{e}_1\mathbf{e}_2 - \mathbf{e}_2\mathbf{e}_1) + v_1u_3(\mathbf{e}_1\mathbf{e}_3 - \mathbf{e}_3\mathbf{e}_1) \\
&\quad + v_2u_3(\mathbf{e}_2\mathbf{e}_3 - \mathbf{e}_3\mathbf{e}_2) + u_1v_2(\mathbf{e}_1\mathbf{e}_2 - \mathbf{e}_2\mathbf{e}_1) \\
&\quad + u_1v_3(\mathbf{e}_1\mathbf{e}_3 - \mathbf{e}_3\mathbf{e}_1) + u_2v_3(\mathbf{e}_2\mathbf{e}_3 - \mathbf{e}_3\mathbf{e}_2)] \\
&= (v_1u_2 - u_1v_2)\mathbf{e}_1\mathbf{e}_2 + (v_1u_3 - u_1v_3)\mathbf{e}_1\mathbf{e}_3 + (v_2u_3 - u_2v_3)\mathbf{e}_2\mathbf{e}_3
\end{aligned}
$$

The latter may be written

$$\mathbf{v} \wedge \mathbf{u} = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3[(v_1u_2 - u_1v_2)\mathbf{e}_3 + (v_3u_1 - v_1u_3)\mathbf{e}_2 + (v_2u_3 - u_2v_3)\mathbf{e}_1]$$

The Gibbs cross product is

$$\mathbf{v} \times \mathbf{u} = \mathbf{e}_1(v_2u_3 - v_3u_2) + \mathbf{e}_2(v_3u_1 - u_3v_1) + \mathbf{e}_3(v_1u_2 - u_1v_2)$$

Therefore,

$$\mathbf{v} \wedge \mathbf{u} = -\mathbf{e}(\mathbf{v} \times \mathbf{u}) \quad \textit{or} \quad \mathbf{e}(\mathbf{v} \wedge \mathbf{u}) = \mathbf{v} \times \mathbf{u} \qquad \mathbf{e} = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 \qquad \mathbf{ee} = -1$$

where $\mathbf{v} \times \mathbf{u}$ is the Gibbs vector product.

Thus, the quantity $\mathbf{e} \equiv \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$ times the wedge product of 2 vectors defines a vector perpendicular to the plane of the 2 vectors. The wedge product of 2 vectors is the sum of 3 bivectors, that is, 3 directed planes[1]. In 4 dimensions, it would be the sum of 6 bivectors or 6 directed planes. Thus Clifford Algebra is a more general geometric

[1] The "direction" of a directed plane is defined by $\mathbf{e}_1 \times \mathbf{e}_2$ where looking down $\mathbf{e}_1$ rotates 90° CC to define a direction perpendicular to $\mathbf{e}_1\mathbf{e}_2$ in the direction of a right hand screw turning from $\mathbf{e}_1$ to $\mathbf{e}_2$, the usual right hand coordinate system.

algebra than vector analysis, that is, the Gibbs algebra since the Gibbs cross product in 4 dimensions and higher is not defined.

Direct Product of (vec)(biv)

$$\begin{aligned}
& \left(v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3\right)\left(B_{12}\mathbf{e}_1\mathbf{e}_2 + B_{31}\mathbf{e}_3\mathbf{e}_1 + B_{23}\mathbf{e}_2\mathbf{e}_3\right) \\
= \; & v_1B_{12}\mathbf{e}_2 - v_1B_{31}\mathbf{e}_3 + v_1B_{23}\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 \\
& -v_2B_{12}\mathbf{e}_1 + v_2B_{31}\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 + v_2B_{23}\mathbf{e}_3 \\
& +v_3B_{12}\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 + v_3B_{31}\mathbf{e}_1 - v_3B_{23}\mathbf{e}_2 \\
= \; & \left(v_3B_{31} - v_2B_{12}\right)\mathbf{e}_1 + \left(v_1B_{12} - v_3B_{23}\right)\mathbf{e}_2 + \left(v_2B_{23} - v_1B_{31}\right)\mathbf{e}_3 \\
& + \left(v_1B_{23} + v_2B_{31} + v_3B_{12}\right)\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 \\
= \; & \mathbf{v}\wedge\mathbf{B} + \mathbf{v}\bullet\mathbf{B} = \mathbf{B}\times\mathbf{v} + \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\left(\mathbf{v}\bullet\mathbf{B}\right)
\end{aligned}$$

3.5 Rotation of Vectors

Clifford Algebra provides a simple operator for rotating vectors. To obtain this operator we first find a procedure for reflecting a vector $\mathbf{b}$ in a plane (mirror) containing another vector $\mathbf{a}$. The mirror is perpendicular to the plane of $\mathbf{a}$ and $\mathbf{b}$. Fig. 3.2a. The inverse of a vector is defined by $\mathbf{a}^{-1}\mathbf{a} = 1$. If $\mathbf{a}$ is a unit vector defined by $\mathbf{aa} = 1$, then $\mathbf{a}^{-1} = \mathbf{a}$. The reflection of $\mathbf{b}$ through $\mathbf{a}$ to obtain $\mathbf{b}'$ is achieved by forming the combination

$$\mathbf{b}^{'} = \mathbf{a}^{-1}\mathbf{ba} = \mathbf{a}^{-1}\mathbf{b}_{\|}\mathbf{a} - \mathbf{a}^{-1}\mathbf{b}_{\perp}\mathbf{a} = \mathbf{a}^{-1}\mathbf{ab}_{\|} + \mathbf{a}^{-1}\mathbf{ab}_{\perp} = \mathbf{b}_{\|} + \mathbf{b}_{\perp}$$

$\mathbf{ab}_{\perp} = -\mathbf{b}_{\perp}\mathbf{a}$ by anticommutativity of orthogonal vectors in Clifford algebra.

To rotate a vector by an angle φ draw 2 vectors $\mathbf{a}$ and $\mathbf{b}$ separated by angle $\varphi/2$ as shown in Fig. 3.2a. $\mathbf{a}$ and $\mathbf{b}$ may be unit vectors. We wish to rotate vector $\mathbf{v}$ into vector $\mathbf{v}'$. Let vector $\mathbf{v}$ make an angle θ_1 with $\mathbf{a}$ where $\theta_1 < \varphi$. Reflect $\mathbf{v}$ through angle θ_1 so that it becomes vector $\mathbf{v}''$. Reflect $\mathbf{v}''$ in $\mathbf{b}$ through angle $\theta_2 = \varphi/2 - \theta_1$. The net reflection of $\mathbf{v}$ is then through an angle $2\theta_1 + 2\left(\varphi/2 - \theta_1\right) = \varphi$. Mathematically, the two reflections are described by

$$\mathbf{v}'' = \mathbf{a}^{-1}\mathbf{va} \tag{3.3}$$

$$\mathbf{v}' = \mathbf{b}^{-1}\mathbf{v}''\mathbf{b} \tag{3.4}$$

Substituting the result of the first reflection into the second reflection

$$\mathbf{v}' = \mathbf{b}^{-1}\mathbf{a}^{-1}\mathbf{vab}$$

$$\mathbf{v}' = \mathbf{bavab} \tag{3.5}$$

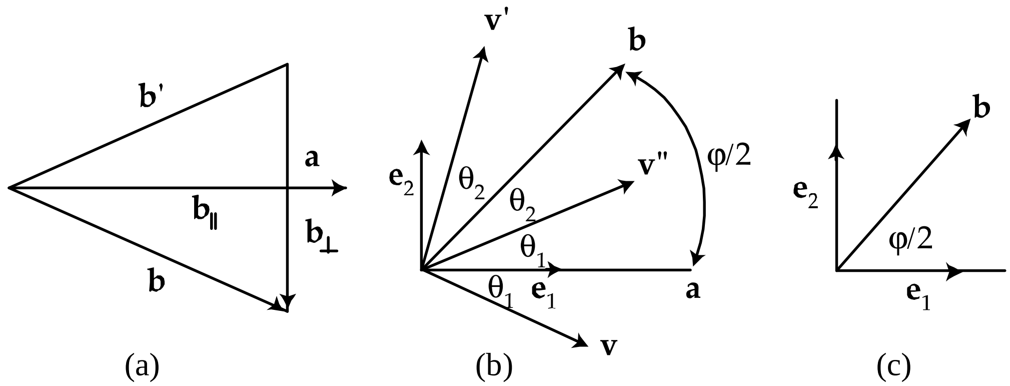

Fig. 3.2

It is convenient to let $\mathbf{a} = \mathbf{e}_1$ and to express $\mathbf{b}$ in terms of $\mathbf{e}_1$ and the unit vector $\mathbf{e}_2$ perpendicular to $\mathbf{e}_1$. Let $\varphi/2$ be the angle between $\mathbf{e}_1$ and $\mathbf{b}$, where φ is the angle through which we wish to rotate the vector $\mathbf{v}$.

Thus,

$$\mathbf{b}_{\|} = \mathbf{e}_1 \cos\varphi/2 \text{ and } \mathbf{b}_{\perp} = \mathbf{e}_2 \sin\varphi/2$$

and therefore,

$$\mathbf{e}_1\mathbf{b} = \mathbf{e}_1 (\mathbf{e}_1 \cos\varphi/2 + \mathbf{e}_2 \sin\varphi/2) = \cos\varphi/2 + \mathbf{e}_1\mathbf{e}_2 \sin\varphi/2 \tag{3.6}$$

which can be written

$$\mathbf{e}_1\mathbf{b} = \mathbf{e}^{\mathbf{e}_1\mathbf{e}_2\varphi/2}, \quad \text{since} \quad (\mathbf{e}_1\mathbf{e}_2)(\mathbf{e}_1\mathbf{e}_2) = -1$$

Likewise,

$$\mathbf{b}\mathbf{e}_1 = (\mathbf{e}_1 \cos\varphi/2 + \mathbf{e}_2 \sin\varphi/2)\,\mathbf{e}_1 = \cos\varphi/2 - \mathbf{e}_1\mathbf{e}_2 \sin\varphi/2 = \mathbf{e}^{-\mathbf{e}_1\mathbf{e}_2\varphi/2} \tag{3.7}$$

Substituting these expressions in Eq. (3.5), we obtain

$$\mathbf{v}' = \mathbf{e}^{-\mathbf{e}_1\mathbf{e}_2\varphi/2}\mathbf{v}\mathbf{e}^{\mathbf{e}_1\mathbf{e}_2\varphi/2} \tag{3.8}$$

Eq. (3.8) specifies a counter clockwise rotation of $\mathbf{v}$ through an angle φ about axis $\mathbf{e}_3$ perpendicular to the plane of $\mathbf{e}_1$ and $\mathbf{e}_2$. To rotate the vector $\mathbf{v}$ clockwise through φ, change φ to $-\varphi$ in the above expressions.

The bilateral operators in Eq. (3.8) may be replaced by the unilateral operator

$$\mathbf{v}' = \mathbf{e}^{-\mathbf{e}_1\mathbf{e}_2\varphi/2}\mathbf{v}\mathbf{e}^{\mathbf{e}_1\mathbf{e}_2\varphi/2} = \mathbf{e}^{-\mathbf{e}_1\mathbf{e}_2\varphi}\mathbf{v} \tag{3.9}$$

when $\mathbf{v}$ lies in the plane of $\mathbf{e}_1\mathbf{e}_2$. Thus, $\mathbf{e}^{-\mathbf{e}_1\mathbf{e}_2\varphi}$ operating left to right on $\mathbf{v}$ rotates it at an angle φ counter-clockwise about $\mathbf{e}_3$. $\mathbf{e}^{\mathbf{e}_1\mathbf{e}_2\varphi}$ operating right to left does the same. To rotate clockwise, change the sign of φ. To justify Eq. (3.9), use Eq. (3.8) to rotate the vector $\mathbf{v} = v_x\mathbf{e}_1 + v_y\mathbf{e}_2$ into $\mathbf{v}' = v_x'\mathbf{e}_1 + v_y'\mathbf{e}_2$

Consider

$$\begin{aligned} v_1\mathbf{e}_1\mathbf{e}^{\mathbf{e}_1\mathbf{e}_2\varphi/2} &= v_1\mathbf{e}_1\left(\cos\varphi/2 + \mathbf{e}_1\mathbf{e}_2\sin\varphi/2\right) = v_1\mathbf{e}_1\left[\cos\varphi/2 + \mathbf{e}_1\mathbf{e}_2\mathbf{e}_1\left(\sin\varphi/2\right)\mathbf{e}_1\right] \\ &= \left(\cos\varphi/2 + \mathbf{e}_1\mathbf{e}_1\mathbf{e}_2\mathbf{e}_1\sin\varphi/2\right)v_1\mathbf{e}_1 = \left(\cos\varphi/2 - \mathbf{e}_1\mathbf{e}_2\sin\varphi/2\right)v_1\mathbf{e}_1 \\ &= \mathbf{e}^{-\mathbf{e}_1\mathbf{e}_2\varphi/2}v_1\mathbf{e}_1 \end{aligned}$$

Likewise
$$v_y\mathbf{e}_2\mathbf{e}^{\mathbf{e}_1\mathbf{e}_2\varphi/2} = \mathbf{e}^{-\mathbf{e}_1\mathbf{e}_2\varphi/2}v_y\mathbf{e}_2$$
So
$$\left(v_x\mathbf{e}_1 + v_y\mathbf{e}_2\right)\mathbf{e}^{\mathbf{e}_1\mathbf{e}_2\varphi/2} = \mathbf{e}^{-\mathbf{e}_1\mathbf{e}_2\varphi/2}\left(v_x\mathbf{e}_1 + v_y\mathbf{e}_2\right)$$

Therefore
$$\begin{aligned} \mathbf{e}^{-\mathbf{e}_1\mathbf{e}_2\varphi/2}\mathbf{v}\mathbf{e}^{\mathbf{e}_1\mathbf{e}_2\varphi/2} &= \mathbf{e}^{-\mathbf{e}_1\mathbf{e}_2\varphi/2}\mathbf{e}^{-\mathbf{e}_1\mathbf{e}_2\varphi/2}\left(v_x\mathbf{e}_1 + v_y\mathbf{e}_2\right) = \mathbf{e}^{-\mathbf{e}_1\mathbf{e}_2\varphi/2}\mathbf{e}^{-\mathbf{e}_1\mathbf{e}_2\varphi/2}\mathbf{v} \\ &= \mathbf{e}^{-\mathbf{e}_1\mathbf{e}_2\varphi}\mathbf{v} = \mathbf{v}' = v_x'\mathbf{e}_1 + v_y'\mathbf{e}_2 \end{aligned}$$

As an example, let us use Eq. (3.9) to rotate counterclockwise through an angle φ a vector $\mathbf{x} = \mathbf{e}_1 a\cos\alpha + \mathbf{e}_2 a\sin\alpha$, making an angle α with respect to the $\mathbf{e}_1$ axis. This will yield two trigonometric addition formulas.

$$\begin{aligned} \mathbf{x}' &= \mathbf{e}^{-\mathbf{e}_1\mathbf{e}_2\varphi}\mathbf{x} = \mathbf{e}^{-\mathbf{e}_1\mathbf{e}_2\varphi}\left(\mathbf{e}_1 a\cos\alpha + \mathbf{e}_2 a\sin\alpha\right) \\ &= \mathbf{e}_1 a\cos\alpha\left(\cos\varphi - \mathbf{e}_1\mathbf{e}_2\sin\varphi\right) + \mathbf{e}_2 a\sin\alpha\left(\cos\varphi - \mathbf{e}_1\mathbf{e}_2\sin\varphi\right) \\ &= a\left[\mathbf{e}_1\left(\cos\alpha\cos\varphi - \sin\alpha\sin\varphi\right) + \mathbf{e}_2\left(\cos\alpha\sin\varphi + \sin\alpha\cos\varphi\right)\right] \\ &= \mathbf{e}_1 a\cos\left(\alpha+\varphi\right) + \mathbf{e}_2 a\sin\left(\alpha+\varphi\right) \end{aligned}$$

Thus

$$\begin{aligned} \cos\left(\alpha+\varphi\right) &= \cos\alpha\cos\varphi - \sin\alpha\sin\varphi \\ \sin\left(\alpha+\varphi\right) &= \cos\alpha\sin\varphi + \sin\alpha\cos\varphi \end{aligned}$$

If $\alpha = 0$, $\mathbf{x}' = a\left(\mathbf{e}_1\cos\varphi + \mathbf{e}_2\sin\varphi\right)$

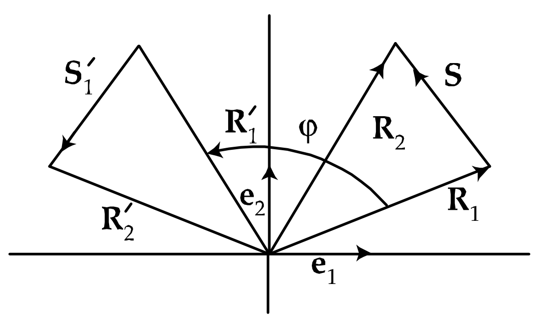

Fig. 3.3

To rotate an arbitrary vector **S** counterclockwise through an angle φ and whose end points are specified by $\mathbf{R}_1$ and $\mathbf{R}_2$, it is necessary to rotate the two vectors defining the end points of **S** through the angle φ. Thus by Fig. 3.3, one has

$$\begin{aligned}
\mathbf{R}_1 + \mathbf{S} &= \mathbf{R}_2 \qquad \mathbf{S} = \mathbf{R}_2 - \mathbf{R}_1 \\
\mathbf{R}_1' + \mathbf{S}' &= \mathbf{R}_2' \qquad \mathbf{S}' = \mathbf{R}_2'' - \mathbf{R}_1' \\
\mathbf{R}_1' &= \mathbf{e}^{-\mathbf{e}_1\mathbf{e}_2\varphi}\mathbf{R}_1 \\
\mathbf{R}_2' &= \mathbf{e}^{-\mathbf{e}_1\mathbf{e}_2\varphi}\mathbf{R}_2 \\
\mathbf{R}_2' - \mathbf{R}_1' &= \mathbf{S}' = \mathbf{e}^{-\mathbf{e}_1\mathbf{e}_2\varphi}(\mathbf{R}_2 - \mathbf{R}_1) = \mathbf{e}^{-\mathbf{e}_1\mathbf{e}_2\varphi}\mathbf{S} \\
\mathbf{S}' &= \mathbf{e}^{-\mathbf{e}_1\mathbf{e}_2\varphi}\mathbf{S}
\end{aligned}$$

To verify that the scalar product is invariant, that is, $\mathbf{S}' \cdot \mathbf{S}' = \mathbf{S} \cdot \mathbf{S}$, form

$$\begin{aligned}
\mathbf{S}' \cdot \mathbf{S}' &= \mathbf{e}^{-\mathbf{e}_1\mathbf{e}_2\varphi}\mathbf{S}\mathbf{e}^{-\mathbf{e}_1\mathbf{e}_2\varphi}\mathbf{S} = \mathbf{e}^{-\mathbf{e}_1\mathbf{e}_2\varphi}\mathbf{e}^{\mathbf{e}_1\mathbf{e}_2\varphi}\mathbf{S} \cdot \mathbf{S} \\
\text{Thus} \qquad \mathbf{S}'^2 &= \mathbf{S}^2
\end{aligned}$$

The rotation of a multivector about $\mathbf{e}_3$ is by definition the rotation about $\mathbf{e}_3$ of the individual vectors that constitute the multivector. To rotate a multivector **M** about $\mathbf{e}_3$ counterclockwise through an angle φ about $\mathbf{e}_3$, form

$$\begin{aligned}
\mathbf{M}' &= \mathbf{L}^{-1}\mathbf{M}\mathbf{L} \\
\mathbf{L} &= \mathbf{e}^{\mathbf{e}_1\mathbf{e}_2\varphi/2}
\end{aligned}$$

It is clear that we can always sandwich $\mathbf{e}^{-\mathbf{e}_1\mathbf{e}_2\varphi/2}\mathbf{e}^{\mathbf{e}_1\mathbf{e}_2\varphi/2} = 1$ between the elements of a multivector. For example, for the rotation of a bivector **vu**

$$\begin{aligned}
\mathbf{e}^{-\mathbf{e}_1\mathbf{e}_2\varphi/2}\mathbf{v}\mathbf{u}\mathbf{e}^{\mathbf{e}_1\mathbf{e}_2\varphi/2} &= \mathbf{e}^{-\mathbf{e}_1\mathbf{e}_2\varphi/2}\mathbf{v}\mathbf{e}^{\mathbf{e}_1\mathbf{e}_2\varphi/2}\mathbf{e}^{-\mathbf{e}_1\mathbf{e}_2\varphi/2}\mathbf{u}\mathbf{e}^{\mathbf{e}_1\mathbf{e}_2\varphi/2} \\
&= \mathbf{v}'\mathbf{u}'
\end{aligned}$$

3.6 Rotation about an Arbitrary Axis n

We now show that the rotation of a vector **v** through an angle φ about an arbitrary axis specified by the unit vector **n** is given by:

$$\mathbf{v}' = \mathbf{e}^{-\mathbf{e}\mathbf{n}\varphi/2}\mathbf{v}\mathbf{e}^{\mathbf{e}\mathbf{n}\varphi/2} \qquad (3.10)$$

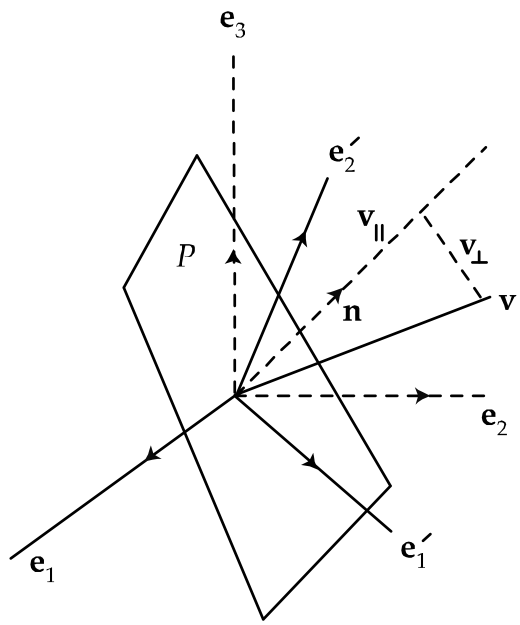

Fig. 3.4

To obtain this result, draw a plane through the origin and perpendicular to $\mathbf{n}$. Decompose $\mathbf{v}$ into a component $\mathbf{v}_\perp$ perpendicular to $\mathbf{n}$ and a component $a\mathbf{n} = \mathbf{v}_\| \mathbf{n}$ parallel to $\mathbf{n}$. Let $\mathbf{e}_1'$ and $\mathbf{e}_2'$ through the origin define two orthogonal axes in the plane. The rotation through an angle φ of the component $\mathbf{u}$ perpendicular to $\mathbf{n}$ is given by

$$\mathbf{v}_\perp' = \mathbf{e}^{-\mathbf{e}_1'\mathbf{e}_2'\varphi/2}\mathbf{v}_\perp \mathbf{e}^{\mathbf{e}_1'\mathbf{e}_2'\varphi/2} \tag{3.11}$$

$\mathbf{e}_1'$ and $\mathbf{e}_2'$ can be specified in terms of direction cosines with respect to the axes $\mathbf{e}_{1,}\mathbf{e}_{2,}\mathbf{e}_3$.

Then

$$\begin{aligned}
\mathbf{e}_1'\mathbf{e}_2' &= (\mathbf{e}_1\cos\alpha_1 + \mathbf{e}_2\cos\beta_1 + \mathbf{e}_3\cos\gamma_1)(\mathbf{e}_1\cos\alpha_2 + \mathbf{e}_2\cos\beta_2 + \mathbf{e}_3\cos\gamma_2) \\
&= \mathbf{e}_1\mathbf{e}_1\cos\alpha_1\cos\alpha_2 + \mathbf{e}_2\mathbf{e}_2\cos\beta_1\cos\beta_2 + \mathbf{e}_3\mathbf{e}_3\cos\gamma_1\cos\gamma_2 \\
&\quad + \mathbf{e}_1\mathbf{e}_2(\cos\alpha_1\cos\beta_2 - \cos\alpha_2\cos\beta_1) + \mathbf{e}_1\mathbf{e}_3(\cos\alpha_1\cos\gamma_2 - \cos\alpha_2\cos\gamma_1) \\
&\quad + \mathbf{e}_2\mathbf{e}_3(\cos\beta_1\cos\gamma_2 - \cos\beta_2\cos\gamma_1)
\end{aligned}$$

Since $\mathbf{e}_1$ and $\mathbf{e}_2$ are orthogonal, the scalar product is zero:

$$\mathbf{e}_1' \cdot \mathbf{e}_2' = \cos\alpha_1\cos\alpha_2 + \cos\beta_1\cos\beta_2 + \cos\gamma_1\cos\gamma_2 = 0$$

The unit vector $\mathbf{n}$ is given by

$$\mathbf{n} = \mathbf{e}_1' \times \mathbf{e}_2' = -\mathbf{e}(\mathbf{e}_1' \wedge \mathbf{e}_2') = \mathbf{e}_1(\cos\beta_1\cos\gamma_2 - \cos\beta_2\cos\gamma_1)$$

$$+\mathbf{e}_2\left(\cos\gamma_1\cos\alpha_2-\cos\alpha_1\cos\gamma_2\right)+\mathbf{e}_3\left(\cos\alpha_1\cos\beta_2-\cos\alpha_2\cos\beta_1\right)$$
$$=\mathbf{e}_1\cos\alpha+\mathbf{e}_2\cos\beta+\mathbf{e}_3\cos\gamma$$

where α, β, and γ are the direction cosines of $\mathbf{n}$. $\mathbf{e}=\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$. The product $\mathbf{e}_1'\mathbf{e}_2'$ may then be written

$$\begin{aligned}\mathbf{e}_1'\mathbf{e}_2' &= \mathbf{e}_1\mathbf{e}_2\cos\gamma+\mathbf{e}_1\mathbf{e}_3\cos\beta+\mathbf{e}_2\mathbf{e}_3\cos\alpha\\ &= \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\left(\mathbf{e}_1\cos\alpha+\mathbf{e}_2\cos\beta+\mathbf{e}_3\cos\gamma\right)\\ &= \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\mathbf{n}=\mathbf{en}\end{aligned}$$

Note that $$\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\mathbf{n} = \mathbf{n}\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$$

Thus, $$\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\mathbf{n} = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\mathbf{n}\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\mathbf{n}=\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\mathbf{n}^2=\left(\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\right)^2\mathbf{n}^2=-1$$

Since, $$\begin{aligned}\mathbf{nn} &= 1\\ \mathbf{nn} &= \left(\mathbf{e}_1\cos\alpha+\mathbf{e}_2\cos\beta+\mathbf{e}_3\cos\gamma\right)\left(\mathbf{e}_1\cos\alpha+\mathbf{e}_2\cos\beta+\mathbf{e}_3\cos\gamma\right)\\ &= \mathbf{e}_1\mathbf{e}_1\cos^2\alpha+\mathbf{e}_2\mathbf{e}_2\cos^2\beta+\mathbf{e}_3\mathbf{e}_3\cos^2\gamma+\left(\mathbf{e}_1\mathbf{e}_2+\mathbf{e}_2\mathbf{e}_1\right)\cos\alpha\cos\beta\\ &\quad+\left(\mathbf{e}_1\mathbf{e}_3+\mathbf{e}_3\mathbf{e}_1\right)\cos\gamma+\left(\mathbf{e}_2\mathbf{e}_3+\mathbf{e}_3\mathbf{e}_2\right)\cos\beta\\ &= \cos^2\alpha+\cos^2\beta+\cos^2\gamma=1\end{aligned}$$

Note that $$\begin{aligned}\mathbf{n} &= \mathbf{e}^{-\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\mathbf{n}\varphi/2}\mathbf{n}\mathbf{e}^{\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\mathbf{n}\varphi/2}\\ &= \left(\cos\frac{\varphi}{2}-\mathbf{en}\sin\frac{\varphi}{2}\right)\mathbf{n}\left(\cos\frac{\varphi}{2}-\mathbf{en}\sin\frac{\varphi}{2}\right)\\ &= \mathbf{n}\left(\cos\frac{\varphi}{2}-\mathbf{en}\sin\frac{\varphi}{2}\right)\left(\cos\frac{\varphi}{2}-\mathbf{en}\sin\frac{\varphi}{2}\right)=\mathbf{n}\end{aligned}$$

Thus, for $$\begin{aligned}a &= \mathbf{v}\cdot\mathbf{n}=a\mathbf{v}_\parallel\\ \mathbf{v}' &= \mathbf{e}^{-\mathbf{en}\varphi/2}\left(a\mathbf{n}+\mathbf{v}_\perp\right)\mathbf{e}^{\mathbf{en}\varphi/2}=\mathbf{e}^{-\mathbf{en}\varphi/2}a\mathbf{n}\mathbf{e}^{\mathbf{en}\boldsymbol{\varphi}/\mathbf{2}}+\mathbf{e}^{\mathbf{en}\boldsymbol{\varphi}/\mathbf{2}}\mathbf{v}_\perp\mathbf{e}^{\mathbf{en}\varphi/2}\\ \mathbf{v}' &= a\mathbf{n}+\mathbf{e}^{-\mathbf{en}\varphi}\mathbf{v}_\perp=a\mathbf{v}_\parallel+\mathbf{e}^{-\mathbf{en}\varphi}\mathbf{v}_\perp\end{aligned} \tag{3.12}$$