2

PERIHELION ADVANCE OF MERCURY

The perihelion of Mercury, that is, the point in its orbit closest to the sun, is observed to advance 5600 seconds of arc per century. The Earth's precession, that is, the changing orientation of the Earth's axis, and the Newtonian gravitational interaction with other planets accounts for all but 43 seconds of this advance (Strom 1987).

Classical Newtonian mechanics predicts all but 43 seconds of the total observed advance of 5600 seconds of arc per century. The missing 43 seconds was a mystery before Einstein. For a planet of mass m traveling at speed v in a circular orbit of radius r, Einstein theory gives $6v^2/c^2$ as a small correction to the gravitational force $\mathbf{F}$.

$$\mathbf{F} = \frac{GMm}{r^2}\left(1 + 6\frac{v^2}{c^2}\right) = \frac{mv^2}{r} \tag{2.1}$$

This correction accounts for the unexplained observed advance of the perihelion of Mercury of 43 seconds of arc per century (Strom 1987). This is an exercise provided by French (1971). We later show that the correction factor $6v^2/c^2$ may be obtained by a method other than Einstein's equations.

2.1 Calculation of the Period of Revolution of Mercury when the Effective Value of G is $G\left(1+6v^2/c^2\right)$

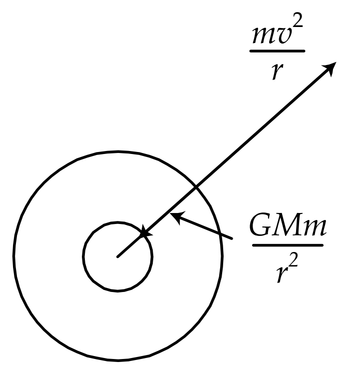

Fig. 2.1

The period of the unperturbed (Newtonian) orbit is obtained from

$$\frac{GMm}{r^2} = \frac{mv^2}{r}$$

$$v^2 = \frac{GM}{r}, \qquad v = \sqrt{\frac{GM}{r}}, \qquad v^2 r = GM \tag{2.2}$$

Time for one orbit is the period T_0

$$T_0 = \frac{2\pi r}{v} = \frac{2\pi r^{3/2}}{\sqrt{GM}} \tag{2.3}$$

$$GM = \frac{(2\pi)^2 r^3}{T_0^2} \tag{2.4}$$

To calculate the period, T, given by the force law, Eq. (2.1), replace G in Eq. (2.3) by $G\left(1+\frac{6v^2}{c^2}\right)$

$$T = \frac{2\pi r^{3/2}}{\sqrt{GM}\sqrt{1+6v^2/c^2}} = T_0\left(1-3v^2/c^2\right) \tag{2.5}$$

From Eq. (2.2) and Eq. (2.4)

$$\frac{v^2}{c^2} = \frac{GM}{c^2 r} = \frac{4\pi^2 r^3}{T_0^2 c^2 r} = \frac{4\pi^2 r^2}{c^2 T_0^2}, \qquad GM = \frac{(2\pi)^2 r^3}{T_0^2} \tag{2.6}$$

Put Eq. (2.6) for v^2/c^2 in Eq. (2.5) to obtain

$$T = T_0 \left(1 - \frac{12\pi^2 r^2}{c^2 T_0^2}\right) \tag{2.7}$$

Calculation of Mercury advance from Eq. (2.7). Without perturbation the angular velocity is:

$$\omega_0 = \frac{2\pi}{T_0}$$

With perturbation the angular velocity is:

$$\begin{aligned} \omega &= \frac{2\pi}{T} = \frac{2\pi}{T_0 (1 - \triangle)} \cong \frac{2\pi (1 + \triangle)}{T_0} \\ \triangle &= 12\pi^2 r^2 / c^2 T_0^2 \end{aligned}$$

Angular distance traveled in time T_0 is:

$$\theta = \omega T_0 = \frac{2\pi (1 + \triangle) T_0}{T_0} = 2\pi (1 + \triangle)$$

Therefore, for each revolution of the planet, it will travel an angle greater by about

$$\triangle\theta = 2\pi\Delta = \frac{24\pi^3 r^2}{c^2 T_0^2} = \frac{24\pi^3 r^2}{c^2} \frac{GM}{4\pi^2 r^3} = \frac{6\pi GM}{c^2 r} \text{ radians per revolution}$$

farther than under the pure Newtonian force. In one century

$$\frac{6\pi GM}{c^2 r} \times N$$

N = number of revolutions per century.

Time for one revolution = 88 days.

$$\begin{aligned} N &= \frac{\text{days in century}}{88 \text{ days}} = \frac{365 \times 100}{88} \\ \frac{\Delta\theta}{\text{century}} &= \frac{6\pi GM}{c^2 r} \frac{36500}{88} \frac{\text{radius}}{\text{century}} = \frac{6\pi GM}{c^2 r} \frac{3.65 \times 10^4}{88} \times \frac{(36)^2 \times 10^3}{2\pi} \frac{\text{sec of arc}}{\text{century}} \\ 1 \text{ radian} &= \frac{360}{2\pi} \text{ degrees} = \frac{360 \times 3600}{2\pi} \text{ seconds of arc} \\ \frac{\Delta\theta}{\text{century}} &= 43 \text{ seconds} \end{aligned}$$

2.2 Relativistic Effect on the Period

Given the following non-relativistic relations, calculate the period of a particle in a gravitational field and then correct it for relativistic velocity.

$$\begin{aligned}\frac{GMm}{r^2} &= \frac{mv^2}{r} \\ v^2 &= \frac{GM}{r} \qquad v = \sqrt{\frac{GM}{r}}\end{aligned} \tag{2.8}$$

The non-relativistic time for one orbit is the period given by

$$T_0 = \frac{2\pi r}{v} = \frac{2\pi r^{3/2}}{\sqrt{GM}} \tag{2.9}$$

Now evaluate the period including the relativistic correction to v, namely $v \to v/\sqrt{1 - v^2/c^2}$. Then the relativistic period is

$$T = \frac{2\pi r\sqrt{1 - v^2/c^2}}{v} \cong T_0 \left(1 - \frac{v^2}{2c^2}\right) \tag{2.10}$$

Consider a period given by

$$T = \frac{2\pi r^{3/2}}{\sqrt{GM}\sqrt{1 + v^2/c^2}} = \frac{2\pi r^{3/2}}{\sqrt{MG\left(1 + v^2/c^2\right)}} = T_0 \left(1 - \frac{v^2}{2c^2}\right) \tag{2.11}$$

that is, change G to $G\left(1 + v^2/c^2\right)$. Eq.(2.11) agrees with Eq.(2.10), which shows that changing G by $1 + v^2/c^2$ gives the relativistically calculated period. Therefore the relativistic correction to v provides one v^2/c^2 of the required $6v^2/c^2$ correction to G to give the correct perihelion advance. Therefore, $5v^2/c^2$ additional corrections to G are needed to agree with the experimental value for the perihelion advance. To adapt the preceding results to gravity, replace q_t^2 by $\left[i\left(c\mathbf{e}_0 + \mathbf{v}\right)\right]^2$ and take the scalar part. No contraction is involved.

$$\left[i\left(c\mathbf{e}_0 + \mathbf{v}\right)\right]^2 = i^2\left(-c^2 + v_t^2\right) = \left(c^2 - v_t^2\right) = c^2\left(1 - v_t^2/c^2\right)$$

For further details, see Chapters 6 and 8.

2.3 Orbital Period in a Schwarzschild metric

$$
\begin{aligned}
v &\rightarrow \frac{v}{\sqrt{1-v^2/c^2}} \rightarrow \frac{v}{\sqrt{1-v^2/c_\varphi^2}} \\
\frac{GMm}{r^2} &= \frac{mv^2}{r} \\
v &= \sqrt{\frac{GM}{r}} \\
T_0 &= \frac{2\pi r}{v}
\end{aligned}
$$

$$
\begin{aligned}
T &= \frac{2\pi r\sqrt{1-v^2/c_\varphi^2}}{v} \cong T_0\left(1-v^2/2c_\varphi^2\right) \\
c_\varphi &= (1-2a)\,c \qquad\qquad a=\frac{v^2}{c^2} \\
\frac{v^2}{2c_\varphi^2} &= \frac{v^2}{2\,(1-2a)^2\,c^2} = \frac{v^2}{2c^2}\frac{1}{1-4a} \\
&= \frac{v^2}{2c^2}\,(1+4a)
\end{aligned}
$$

$$
\begin{aligned}
T &= T_0\left(1-\frac{v^2}{2c^2\,(1-4a)}\right) = T_0\left[1-\frac{v^2}{2c^2}\,(1+4a)\right] \\
&= T_0\left[1-\frac{v^2}{2c^2}\left(1+\frac{4v^2}{c^2}\right)\right] = T_0\left[1-\frac{v^2}{2c^2}+\frac{3v^4}{c^4}\right]
\end{aligned}
$$

The third term is a negligible correction to the relativistic correction. Therefore we can ignore the Schwarzschild metric relativistic correction to the relativistic correction in flat space.

2.4 Summary of the Sources of v^2/c^2 to Obtain a Total Value of $6v^2/c^2$ Seconds of Arc per Century for the Advance of the Perihelion of Mercury

These additional corrections are as follows:

1. A special relativity correction contributes $\frac{v^2}{c^2}$ to G. In other words, it reduces the requirement of $\frac{6v^2}{c^2}$ to G to a requirement of $\frac{5v^2}{c^2}$.

2. A non-commutative correction is $\frac{2v^2}{c^2}$. It includes a torque density contribution. Eqs. (8.15), (8.85), (8.86). This is a new force, item 1 below.

3. A classical correction following Page and Adams (1945) is $\frac{3v^2}{c^2}$. For the linear momentum correction portion of this term see Eq. (10.26). For the associated torque correction, see Eq. (10.38).

4. A Schwarzschild metric correction is $-\frac{v^2}{c^2}$. Eq. (7.24).

5. The $\epsilon_0\left(\mathbf{E}_t \times \mathbf{B}_t\right)$ correction is $\frac{v^2}{c^2}$. Section 11.1

The total of all corrections to G is:

Non-commutative	Page & Adams (1945)	Schwarzschild	Contribution of $\epsilon_0\left(\mathbf{E}_t \times \mathbf{B}_t\right)$,
	(classical)	(curved space)	
1.	2.	3.	4.
$\frac{2v^2}{c^2}$	$\frac{3v^2}{c^2}$	$-\frac{v^2}{c^2}$	$\frac{v^2}{c^2}$

Total correction : $\frac{5v^2}{c^2}$

The gravitational field described in the following work follows from electromagnetism and has a similar structure. Therefore the resulting gravitational field may be quantized as it is for the electromagnetic field.