# 16

# SCHWARZCHILD METRIC

## 16.1 Derivation of the Schwarzschild Metric

In flat 3-space, the metric $ds^2$ specifies the infinitesimal distance between two points. In Cartesian coordinates , it is $ds^2 = dx^2 + dy^2 + dz^2$ . In spherical coordinates, it is $ds^2 = dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\varphi^2\right)$ . To describe a curved space, a curvature coefficient may be introduced in front of $dr^2$ giving $ds^2 = n\left(r\right)dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right)$ . In non-relativistic physics, the force between two charges or masses and the associated potential may be derived from the form of the metric characterizing the space. It is based on the assumption that field strength is associated with the spread in the density of geodesics originating at a source. Geodesic density or field strength may be obtained from the radially symmetric solution to the Laplace equation that is derived from the space metric as will be illustrated. Field strength may also be deduced from flux conservation, that is, geodesic conservation.

We next apply the same procedure to a space-time metric. In this case $ds^2$, now called the interval, is no longer simply spatial but has the form:

$$ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2$$

The following shows some gravitational relations in a flat space metric compared with the same quantities with a curved space. $\phi$ = potential of unit mass.

Force on a unit mass is negative.

$$\begin{aligned} f &= -\frac{GM}{r^2} = -\frac{d\phi}{dr} \\ d\phi &= \frac{GM}{r^2} dr \end{aligned}$$

| Curved space | Flat space metric |
|---|---|
| $d\phi = n^{1/2}(r)\frac{GM}{r^2}dr$ | $d\phi = \frac{GM}{r^2}dr$ |
| $f = -\frac{\partial\phi}{\partial r} = -GM\left(\frac{n^{1/2}}{r^2}\right)$ | $f = -\frac{\partial\phi}{\partial r} = -\frac{\partial}{\partial r}\left(\frac{GM}{r}\right) = -\frac{GM}{r^2}$ |
| $\int_r^\infty d\phi = GM\int_r^\infty \frac{n^{1/2}dr}{r^3}$ | $\int_r^\infty d\phi = GM\int_r^\infty \frac{1}{r^2}dr$ |
| $\phi(\infty) - \phi(r) = -GM\int_r^\infty \frac{n^{1/2}(r)}{r^2}dr$ | $-\left[\frac{GM}{r}\right]_r^\infty = \phi(\infty) - \phi(r) = \frac{GM}{r}$ |
| $\phi(\infty) = 0$ | $\phi(\infty) = 0$ |
| $\phi(r) = GM\int_r^\infty \frac{n^{1/2}(r)}{r^2}dr$ | $\phi(r) = -\frac{GM}{r}$ |

The metric describing the distance between two points in spherical coordinates is

$$ds^2 = dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right) \tag{16.1}$$

It yields the Newtonian potential $-GM/r$ by the following argument. The Laplace equation associated with Eq. (16.1) is

$$\nabla^2\phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) + L^2(3)\,\phi = 0 \tag{16.2}$$

$L^2(3)$ represents the angular portion. The flux conserving potential for flat space is the spherically symmetric solution to the equation

$$r^2\frac{\partial\phi}{\partial r} = \text{const} = GM. \tag{16.3}$$

$GM/r^2$ is the force, in this case attractive. Work to go from $r$ to $\infty$ is:

$$\begin{aligned} \phi(\infty) - \phi(r) &= \int_r^\infty \frac{GM}{r^2}dr = \left[-\frac{GM}{r}\right]_r^\infty = \frac{GM}{r} \\ \phi(r) &= -\frac{GM}{r} \end{aligned} \tag{16.4}$$

Thus when $\phi(\infty) = 0$ we obtain the Newtonian potential of a unit mass (potential at $\infty$ is zero)

$$\phi(r) = -\frac{GM}{r} \tag{16.5}$$

Now modify the potential by introducing a radially symmetric curvature in the metric by placing a function $n(r)$ in front of $dr^2$. Then in polar coordinates

$$ds^2 = n(r)\, dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\varphi^2\right) \tag{16.6}$$

The associated Laplace equation is

$$\nabla^2\phi = \frac{1}{r^2 n^{1/2}} \frac{\partial}{\partial r}\left(\frac{r^2}{n^{1/2}} \frac{\partial \phi}{\partial r}\right) + L^2(3)\,\phi = 0. \tag{16.7}$$

The force field intrinsic to the metric is obtained from the spherically symmetric solution, that is putting $L^2(3) = 0$.

$$\frac{r^2}{n^{1/2}(r)} \frac{d\phi}{dr} = \text{const} = GM \tag{16.8}$$

The attractive force field $F_{GM}$ in the radial direction on a particle of unit mass is

$$F_{GM} = f = -\frac{d\phi}{dr} = -\frac{GMn^{1/2}(r)}{r^2} = -\frac{\partial}{\partial r}\left(\frac{-GMn^{1/2}(r)}{r}\right) = GM\frac{\partial}{\partial r}\left(\frac{n^{1/2}}{r}\right) \tag{16.9}$$

In Eq. (16.9), let

$$n(r) = \frac{1}{\left(1 - \dfrac{2GM}{c^2 r}\right)} \tag{16.10}$$

Then

$$F_{GM} = f = \frac{d\phi}{dr} = \frac{-n^{1/2}(r)\,GM}{r^2} = -\frac{GM}{r^2\left(1 - \dfrac{2GM}{c^2 r}\right)^{1/2}} \tag{16.11}$$

Now evaluate from Eq. (16.9)

$$\phi(\infty) - \phi(r) = \int_r^\infty \frac{GM dr}{r^2\left(1 - \dfrac{2GM}{c^2 r}\right)^{1/2}} = \left[c^2\left(1 - \frac{2GM}{c^2 r}\right)^{1/2}\right]_r^\infty$$

$$= \; -c^2\left[\frac{1}{(1 - 2GM/c^2 r)^{1/2}} \frac{1}{2}\left(-\frac{2GM}{c^2} \frac{d}{dr} \frac{1}{r}\right)\right] = \frac{1}{(1 - 2GM/c^2 r)^{1/2}} GM\left[-\frac{1}{r^2}\right]$$

Let $\phi(\infty) = 0$ so that

$$\phi(r) = c^2 \left(1 - \left(1 - 2GM/c^2 r\right)^{1/2}\right) \tag{16.12}$$

Expanding[1] Eq. (16.12) we obtain

$$\begin{aligned}
\phi(r) &= c^2 \left[1 - \left(1 - \frac{2GM}{c^2 r}\right)^{1/2}\right] \\
&= c^2 \left[1 - \left(1 - \frac{GM}{c^2 r} - \frac{1}{2}\left(\frac{GM}{c^2 r}\right)^2 - \frac{1}{16}\left(\frac{GM}{c^2 r}\right)^3 - \dots\right)\right] \\
&= \frac{GM}{r} + \frac{1}{2c^2}\left(\frac{GM}{r}\right)^2 + \frac{1}{c^4}\frac{(GM)^3}{16 r^3} \qquad (16.13) \\
\text{From } \frac{MmG}{r^2} &= \frac{mv^2}{r} \rightarrow \frac{MG}{r} = v^2 \rightarrow \frac{MG}{c^2 r} = \frac{v^2}{c^2} \qquad (16.14) \\
\phi(r) &= \frac{GM}{r} \\
f &= -\frac{\partial \phi}{\partial r} = -\frac{\partial}{\partial r}\left(-\frac{GM}{r}\right) = -\frac{GM}{r^2}
\end{aligned}$$

Thus the form of the curvature coefficient given in Eq. (16.10) yields the correct flat space potential for large $r$.

$$(1-x)^{-1/2} = 1 + \frac{x}{2} + \frac{3}{8}x^2 + \frac{5}{16}x^3$$

$$\left(1 - \frac{2GM}{c^2 r}\right)^{-1/2} = 1 + \frac{v^2}{c^2} + \frac{3}{2}\left(\frac{v^2}{c^2}\right)^2 + \frac{5}{2}\left(\frac{v^2}{c^2}\right)^3 \dots$$

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[1]

$$(a+b)^n = a^n + \frac{na^{n-1}b}{1!} + \frac{n(n-1)a^{n-1}b^2}{2!} + \frac{n(n-1)(n-2)a^{n-2}b^3}{3!} + \dots$$

$$\begin{aligned}
(1-x)^n &= 1 - nx + \frac{n(n-1)x^2}{2!} - \frac{n(n-1)(n-2)x^3}{3!} + \dots \qquad \frac{GM}{c^2 r} = \frac{v^2}{c^2} \\
(1-x)^{1/2} &= 1 - \frac{x}{2} + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)x^2}{2!} - \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)x^3}{3!} \dots \\
\left(1 - \frac{2GM}{c^2 r}\right)^{1/2} &= 1 - \frac{x}{2} - \frac{x^2}{8} - \frac{3x^3}{48} \dots = 1 - \frac{GM}{c^2 r} + \frac{4(GM)^2}{8c^4 r^2} - \frac{8(GM)^3}{48 c^6 r^3} \\
\left(1 - \frac{2GM}{c^2 r}\right)^{1/2} &= 1 - \frac{GM}{c^2 r} + \frac{1}{2}\left(\frac{GM}{c^2 r}\right)^2 - \frac{1}{6}\left(\frac{GM}{c^2 r}\right)^3 \dots = 1 - \frac{v^2}{c^2} + \frac{1}{2}\left(\frac{v^2}{c^2}\right)^2 - \frac{1}{6}\left(\frac{v^2}{c^2}\right)^3 ..
\end{aligned}$$

The radial force field $F_{GM}$ is from Eq. (16.11)

$$\begin{aligned} F_{GM} &= \frac{\partial\phi}{\partial r} = -\frac{GM}{r^2\left(1-\dfrac{2GM}{c^2r}\right)^{1/2}} = -\frac{GM}{r^2}\left(1-\frac{2GM}{c^2r}\right)^{-1/2} \\ F_{GM} &\cong -\frac{GM}{r^2}\left(1+\frac{GM}{c^2r}+\frac{1}{2}\left(\frac{GM}{c^2r}\right)^2+\frac{5}{2}\left(\frac{GM}{c^2r}\right)^3...\right) \end{aligned} \tag{16.15}$$

From $\dfrac{GM}{c^2r}=\dfrac{v^2}{c^2}$

$$\frac{\partial\phi}{\partial r} = F_{GM} \cong -\frac{GM}{r^2}\left[1+\frac{v^2}{c^2}+\frac{1}{2}\left(\frac{v^2}{c^2}\right)^2+\frac{5}{2}\left(\frac{v^2}{c^2}\right)^3\right]$$

Thus the curved space metric gives an effective force

$$F_{GM} \simeq -\frac{GM}{r^2}\left(1+\frac{v^2}{c^2}\right) = -\frac{G\left(1+v^2/c^2\right)M}{r^2} \tag{16.16}$$

## 16.2 Space-time Schwarzschild Metric

Starting with the spatial metric we concluded from the flux conserving condition that $n(r) = 1/\left(1-2GM/c^2r\right)$. The space-time metric may be obtained by requiring volume invariance as we go from the space metric to the space time metric. The volume element in general is

$$dV = \left(g_1g_2—g_n\right)^{1/2} du_1...du_n \tag{16.17}$$

The volume element for the equation

$$ds^2 = dr^2 + r^2\left[(d\theta)^2+\sin^2\theta\,(d\varphi)^2\right] - c^2\,(dt)^2 \tag{16.18}$$

is

$$dV = r^2\sin\theta d\theta d\varphi c dt \tag{16.19}$$

For the space-time metric

$$ds^2 = n(r)\,(dr)^2 + r^2\left[(d\theta)^2+\sin^2\theta d\theta\,(d\varphi)^2\right] - c^2m(r)\,(dt)^2 \tag{16.20}$$

where the coefficient $m(r)$ is introduced to conserve volume as we go from flat space to the Schwarzschild metric. The space-time volume element is

$$dV = n(r)\,m(r)\,r^2\sin\theta d\theta d\varphi c dt \tag{16.21}$$

For Eqs. (16.19) and (16.21) to be the same $m(r) = 1/n(r)$. "Volume" is conserved.

Equation (16.8) may also be derived as follows. Let the total flux emanating from $m$ be $4Gm$. Then for flux conservation in a metric

$$ds^2 = g_1 (du_1)^2 + g_2 (du_2)^2 + g_3 (du_3)^2 \tag{16.22}$$

one can write for the total flux $F$ where $f$ is flux density, and $dS$ is the element of surface area.

$$F = \int f dS = f \int \left(\frac{g}{g_1}\right)^{1/2} du_2 du_3 = f r^2 \int_0^\pi \sin\theta d\theta \int_0^\pi d\varphi = 4Gm \tag{16.23}$$

so that $f = Gm/r^2$.

## 16.3 Schwarzschild Metric in Curved Space. 3 Dimensional Hypersphere

The metric for a 3-dimensional hypersphere is

$$ds^2 = R^2 (d\theta_2)^2 + R^2 \sin^2\theta_2 \left[\left((d\theta_1)^2 + \sin^2\theta_1 (d\varphi)^2\right)\right] \tag{16.24}$$

The space-time metric is

$$ds^2 = R^2 (d\theta_2)^2 + R^2 \sin^2\theta_2 \left[(d\theta_1)^2 + \sin^2\theta_1^2 (d\varphi)^2\right] - c^2 (dt)^2 \tag{16.25}$$

"Curve" the metric of Eq. (16.25) locally by introducing the factor $n(\theta_2)$ so that

$$ds^2 = n(\theta_2) R^2 (d\theta_2)^2 + R^2 \sin^2\theta_2 \left[(d\theta_1)^2 + \sin^2\theta_1 (d\varphi)^2\right] \tag{16.26}$$

from which

$$\begin{aligned} g^1 &= 1/R^2 n(\theta_2), \qquad g^2 = 1/R^2 \sin^2\theta_2, \qquad g^3 = 1/R^2 \sin^2\theta_2 \sin^2\theta_1 \\ g &= R^3 n^{1/2}(\theta_2) \sin^2\theta_2 \sin\theta_1, \end{aligned}$$

The associated Laplace equation is

$$\nabla^2\phi = \frac{1}{R^2 n^{1/2} \sin^2\theta_2}\left[\frac{\partial}{\partial\theta_2}\left(\frac{\sin^2\theta_2}{n^{1/2}}\frac{\partial\varphi}{\partial\theta_2}\right)\right] + \frac{L^2(3)}{R^2\sin^2\theta_2} = 0 \tag{16.27}$$

The force field intrinsic to the metric is obtained from the $\theta_2$ symmetric solution $(L^2(3) = 0)$

$$\frac{\partial}{\partial\theta_2}\left(\frac{\sin^2\theta_2}{n^{1/2}}\frac{\partial\phi}{R\partial\theta_2}\right) = \text{const} = -Gm, \quad \text{so that} \quad \frac{\partial\phi}{\partial\theta_2} = -\frac{Gmn^{1/2}(\theta_2)}{R\sin^2\theta} \tag{16.28}$$

Equation (16.26) gives the "Newtonian" potential, that is, the hypersphere potential $\phi = (Gm \cot \theta_2)/R$ for large $\theta_2 \to \pi/2$ if $n(\theta_2) = 1/(1 - 2mG \cot \theta_2 / c^2 R)$ since then

$$\frac{\partial \phi}{\partial \theta_2} = \frac{Gm}{\sin^2 \theta_2 \left(1 - \dfrac{2Gm \cot \theta_2}{c^2}\right)^{1/2}} \tag{16.29}$$

Integrating

$$\phi(\pi/2) - \phi(\theta_2) = c^2 \left[\left(1 - \frac{2Gm \cot \theta_2}{c^2}\right)^{1/2}\right]_{\theta_2}^{\pi/2} = c^2 \left[1 - \left(1 - \frac{2Gm \cot \theta_2}{c^2}\right)^{1/2}\right] \tag{16.30}$$

Let $\phi(\pi/2) = 0$, so that

$$\phi(\theta_2) = -c^2 \left[1 - \left(1 - \frac{2Gm \cot \theta_2}{c^2}\right)^{1/2}\right]$$

Adding twice this to the coefficient of $dt^2$ in Eq. (16.25) leads to the Lagrangian

$$\begin{aligned} 2L &= n(\theta_2) R^2 \dot{\theta}_2^2 + R^2 \sin^2 \theta_2 \left(\dot{\theta}_1^2 + \sin^2 \theta_1 \dot{\varphi}^2\right) \\ &\quad + 2c^2 \left[1 - \left(1 - \frac{2Gm \cot \theta_2}{c^2}\right)^{1/2}\right] - c^2 \end{aligned} \tag{16.31}$$

Expand twice the potential

$$\begin{aligned} 2c^2 \left[1 - \left(1 - \frac{2Gm \cot \theta_2}{c^2 r}\right)^{1/2}\right] - c^2 &= \frac{2Gm \cot \theta_2}{R} - c^2 + \frac{1}{c^2}\left(\frac{Gm \cot \theta_2}{R}\right)^2 \\ &\quad + \frac{4}{3c^4}\left(\frac{Gm \cot \theta_2}{R}\right)^3 \end{aligned} \tag{16.32}$$

Twice the Lagrangian is

$$2L = \frac{R^2 \dot{\theta}_2^2}{\left(1 - \dfrac{2Gm \cot \theta_2}{c^2 R}\right)} + R^2 \sin^2 \theta_2 \left(\dot{\theta}_1^2 + \sin^2 \theta_1 \dot{\varphi}^2\right) - c^2 \left(1 - \frac{2Gm \cot \theta_2}{c^2 R}\right) \tag{16.33}$$

Multiply by $dt^2$ to obtain the free particle Lagrangian or the Schwarzschild metric

$$\begin{aligned} 2Ldt^2 &= ds^2 = \frac{R^2\left(d\dot{\theta}_2\right)^2}{1-\dfrac{2Gm\cot\theta_2}{c^2R}} + R^2\sin^2\theta_2\left[(d\theta_1)^2+\sin^2\theta_1\,(d\varphi)^2\right] \\ &\quad -\left(1-\frac{Gm\cot\theta_2}{c^2R}\right)c^2dt^2 \end{aligned} \tag{16.34}$$

One may proceed in the same way to obtain the Schwarzschild metric in a 3-dimensional space of negative curvature. The result will be the same as Eq. (16.34) with $\cot\theta_2$ replaced by $\coth\theta_2$. The procedure gives corresponding Schwarzschild metrics in higher dimensional curved spaces.

## 16.4 Comments on the de Sitter Metric

Eq. (16.8) has the form

$$d\phi = \frac{Cn^{1/2}(r)}{r^2}dr \tag{16.35}$$

If $C = 1$ and $n = (1 - r^2/R^2)^{1/2}$ the latter being the coefficient of $dr^2$ in the spatial part of the de Sitter metric

$$ds^2 = \frac{dr^2}{1-r^2/R^2} + r^2d\theta_1^2 + r^2\sin^2\theta_1 d\varphi^2 \tag{16.36}$$

Then, from Laplace's equation

$$d\phi = \frac{(1-r^2/R^2)^{1/2}}{r^2}dr \tag{16.37}$$

Let $r = R\sin\theta_2$,

$$dr = R\cos\theta_2 d\theta_2, \qquad \left(1-r^2/R^2\right)^{1/2} = \cos\theta_2 \tag{16.38}$$

$$\phi = \int_0^{\theta_2}\frac{\cos^2\theta_2 d\theta_2}{R\sin^2\theta} = \frac{\cot\theta_2}{R} \tag{16.39}$$

Thus the flux argument leads to a potential $\cot\theta_2$, which is understandable since Eq. (16.36) is the line element for a 3-dimensional hypersphere. Explicitly, if the substitutions of Eq. (16.38) are placed in Eq. (16.36), $ds^2$ becomes

$$ds^2 = R^2\left(d\theta_2^2 + \sin^2\theta_2 d\theta_1^2 + \sin^2\theta_1 d\theta^2\right) \tag{16.40}$$

which leads to the potential $\phi = (\cot\theta_2)/R$, where $\theta_2 = \arccos\left(1 - r^2/R^2\right)^{1/2}$. For large $R$ and/or small $\theta_2$, $\phi = 1/r$ so the potential associated with Eq. (16.36) also goes over to the Coulomb potential, and when the metric given by Eq. (16.40) becomes flat, that is, when $Rd\theta_2 = dr$, $R\sin\theta_2 = r$.

Thus, the de Sitter metric as well as the Schwarzschild metric are possible solutions for Eq. (16.35). The spatial part of the de Sitter metric yields a potential that differs from the Coulomb potential for large $r$ and goes over to the Coulomb potential for small $r$. This is opposite to the behavior required to properly explain the behavior of the orbit of Mercury.