14
VERIFICATION THAT THE NON-COMMUTATIVE FIELD FORCE IS ZERO

In this chapter, we repeat in a different way some of the results presented in the text.

14.1 Vector Forces Associated with the Transport of the Multivectors of Space-time Algebra

In the following, we will employ two non-commuting space-time algebras. The transport or carrier quantities $\mathbf{V}_s$, $\mathbf{V}_t$, $\mathbf{r}$, and the gradient operator $\Box$ plus the forces $\mathbf{F}_{ts}$ and $\mathbf{F}_{st}$ acting on $q_t\mathbf{V}_t$ and $q_s\mathbf{V}_s$, derived therefrom will be written in heavy case letters. The transported multivectors involved in the interaction will be written in standard non-heavy case letters. Any pair of the non-commuting transported multivectors in the space-time property algebra will be represented by the letters q_s, q_t. This notation more closely resembles standard electromagnetism. q_s is transported with velocity $\mathbf{V}_s$, and q_t is transported with velocity $\mathbf{V}_t$. $\mathbf{V}_s$ and $\mathbf{V}_t$ are space-time four-vectors.

In general q_s and q_t may be different kinds of multivectors, for example, q_s a vector and q_t a bivector. This can be indicated by placing a prime on one or the other. In the present discussion, we regard q_s and q_t as like multivectors, that is, both vectors, both bivectors, etc. The interaction of different kinds of multivectors can be treated separately.

14.2 Indirect Force

Velocity four-vectors as written below represent physical source currents $q_s\mathbf{V}_s$ and $q_t\mathbf{V}_t$ when multiplied by charges q_s and q_t.

$$\begin{aligned}\mathbf{V}_s &= \gamma_s\left(c\mathbf{e}_0 + v_{sx}\mathbf{e}_1 + v_{sy}\mathbf{e}_2 + v_{sz}\mathbf{e}_3\right) = \gamma_s\left(c\mathbf{e}_0 + \mathbf{v}_s\right)\\ \mathbf{V}_t &= \gamma_t\left(c\mathbf{e}_0 + v_{tx}\mathbf{e}_1 + v_{ty}\mathbf{e}_2 + v_{tz}\mathbf{e}_3\right) = \gamma_t\left(c\mathbf{e}_0 + \mathbf{v}_t\right)\end{aligned}$$

A trivector will appear in the equations and may be written

$$T = T^{012}\mathbf{e}_0\mathbf{e}_1\mathbf{e}_2 + T^{031}\mathbf{e}_0\mathbf{e}_3\mathbf{e}_1 + T^{023}\mathbf{e}_0\mathbf{e}_2\mathbf{e}_3 + T^{123}\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$$

$$\mathbf{e}_5 = \mathbf{e}_0\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 \qquad \mathbf{e}_5\mathbf{e}_5 = -1$$

$$\begin{aligned}T &= -\mathbf{e}_5\mathbf{e}_5\left(T^{012}\mathbf{e}_0\mathbf{e}_1\mathbf{e}_2 + T^{031}\mathbf{e}_0\mathbf{e}_3\mathbf{e}_1 + T^{023}\mathbf{e}_0\mathbf{e}_2\mathbf{e}_3 + T^{123}\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\right)\\ &= \mathbf{e}_5\left(u^0\mathbf{e}_0 + u^1\mathbf{e}_1 + u^2\mathbf{e}_2 + u^3\mathbf{e}_3\right) = \mathrm{Triv}\end{aligned}$$

$$u^0 = T^{123}, \qquad u^1 = T^{023}, \qquad u^2 = T^{031}, \qquad u^3 = T^{012}\text{since } \mathbf{e}_5\mathbf{e}_5 = -1$$

$$q_s\overset{\curvearrowleft}{\mathbf{F}_s\Box} = \mu_0 q_s\mathbf{V}_s \tag{14.1}$$

and

$$q_t\overset{\curvearrowleft}{\mathbf{F}_t\Box} = \mu_0 q_t\mathbf{V}_t \tag{14.2}$$

where

$$\mathbf{F}_s = \frac{E_{sx}}{c}\mathbf{e}_0\mathbf{e}_1 + \frac{E_{sy}}{c}\mathbf{e}_0\mathbf{e}_2 + \frac{E_{sz}}{c}\mathbf{e}_0\mathbf{e}_3 + B_{sz}\mathbf{e}_1\mathbf{e}_2 + B_{sy}\mathbf{e}_3\mathbf{e}_1 + B_{sx}\mathbf{e}_2\mathbf{e}_3 \tag{14.3}$$

from which we obtain

$$\begin{aligned}\mathbf{F}_s\Box &= \mathbf{e}_0\mathrm{div}\frac{\mathbf{E}_s}{c} - \frac{\partial\mathbf{E}_s}{c^2\partial t} + \mathrm{curl}\mathbf{B}_s + \mathbf{e}_5\left(-\mathbf{e}_0\mathrm{div}\mathbf{B}_s + \frac{\partial\mathbf{B}_s}{c\partial t} + \mathrm{curl}\frac{\mathbf{E}_s}{c}\right) \\ &= \mu_0\mathbf{V}_s\end{aligned} \tag{14.4}$$

$\Box \bullet \mathbf{A}_s = 0$ is the Lorentz condition that is required to yield Maxwell's equations that empirically are found to describe nature.

In a multivector equation, that is, an equation containing several multivectors, equate like multivectors. In Eq. (14.4), the trivector term is composed of the field quantities:

$$\mu_0 q_s\mathbf{e}_5\left(-\mathbf{e}_0\mathrm{div}\mathbf{B}_s + \frac{\partial\mathbf{B}_s}{c\partial t} + \frac{1}{c}\mathrm{curl}\mathbf{E}_s\right) = 0 \tag{14.5}$$

while for the vector part

$$\mathbf{e}_0 \mathrm{div}\frac{\mathbf{E}_s}{c} - \frac{\partial \mathbf{E}_s}{c^2 \partial t} + \mathrm{curl}\mathbf{B}_s = \mu_o \mathbf{V}_s \tag{14.6}$$

Putting div $\mathbf{B}_s$ equal to zero, and therefore Eq. (14.5) gives the two Maxwell equations:

$$\mathrm{curl}\ \mathbf{E}_s = -\frac{\partial \mathbf{B}_s}{\partial t} \tag{14.7}$$

$$\mathrm{div}\ \mathbf{B}_s = 0 \tag{14.8}$$

Put equal to zero, Eq. (14.6) gives the second two Maxwell Equations:

$$\mathrm{curl}\ \mathbf{B}_s = \frac{1}{c^2}\frac{\partial \mathbf{E}_s}{\partial t} \tag{14.9}$$

$$\mathrm{div}\ \frac{\mathbf{E}_s}{c} = 0 \tag{14.10}$$

Add Eqs. (14.1) and (14.2) to obtain

$$(q_s\mathbf{F}_s + q_t\mathbf{F}_t)\overset{\curvearrowleft}{\square} = \mu_0\left[q_s\mathbf{V}_s + q_t\mathbf{V}_t\right] \tag{14.11}$$

$$\overset{\curvearrowright}{\square}(q_s\mathbf{F}_s + q_t\mathbf{F}_t) = -\mu_0\left[q_s\mathbf{V}_s + q_t\mathbf{V}_t\right] \tag{14.12}$$

Thus, $\square$ applied left to right on $\mathbf{F}$ differs in sign from $\square$ applied right to left on $\mathbf{F}$.

Now multiply Eq.(14.11) on the right by $(q_s\mathbf{F}_s + q_t\mathbf{F}_t)$

$$(q_s\mathbf{F}_s + q_t\mathbf{F}_t)\overset{\curvearrowleft}{\square}(q_s\mathbf{F}_s + q_t\mathbf{F}_t) = \mu_0\left[(q_s\mathbf{V}_s + q_t\mathbf{V}_t)(q_s\mathbf{F}_s + q_t\mathbf{F}_t)\right] \tag{14.13}$$

Multiply Eq.(14.12) on the left by $(q_s\mathbf{F}_s + q_t\mathbf{F}_t)$

$$(q_s\mathbf{F}_s + q_t\mathbf{F}_t)\overset{\curvearrowright}{\square}(q_s\mathbf{F}_s + q_t\mathbf{F}_t) = -\mu_0\left[(q_s\mathbf{F}_s + q_t\mathbf{F}_t)(q_s\mathbf{V}_s + q_t\mathbf{V}_t)\right] \tag{14.14}$$

By Eq. (14.1)

$$\mathbf{F}_s\overset{\curvearrowleft}{\square} = \mu_0 q_s\mathbf{V}_s \qquad \overset{\curvearrowright}{\square}\mathbf{F}_s = -\mu_0 q_s\mathbf{V}_s \tag{14.15}$$

Multiply the two equations in (14.15) on the right and left respectively by $\mathbf{F}_s$. Then add to obtain Eq. (14.16)

$$\mathbf{F}_\mathbf{s}\overset{\curvearrowleft}{\square}\mathbf{F}_s = \mu_0 q_s\mathbf{V}_s\mathbf{F}_s \qquad \mathbf{F}_s\overset{\curvearrowright}{\square}\mathbf{F}_s = -\mu_0 q_s\mathbf{F}_s\mathbf{V}_s$$

$$\left(\mathbf{F}_s\overset{\curvearrowleft}{\square}\mathbf{F}_s + \mathbf{F}_s\overset{\curvearrowright}{\square}\mathbf{F}_s\right)/2 = -\mu_0 q_s\left(\mathbf{F}_s\mathbf{V}_s - \mathbf{V}_s\mathbf{F}_s\right)/2 \tag{14.16}$$

In the following q_s and q_t are regarded as multivectors of like kind. In this case $(q_s q_t + q_t q_s)/2$ is always a scalar and $(q_s q_t - q_t q_s)/2$ is always a bivector.

Add Eq.(14.13) and Eq.(14.14), treating q_s and $q_t$as scalars or labels. After doing so, we then regard the final expression, Eq. (14.17), as a postulate wherein q_s and q_t belong to a space-time algebra that commutes with the algebra used to define $\mathbf{F}_s$, $\mathbf{F}_t$, $\mathbf{V}_s$, $\mathbf{V}_t$ and $\Box$ and that describe the "bare" field.

$$\begin{aligned}
& q_s q_s \overset{\curvearrowleft}{\mathbf{F}_s\Box}\mathbf{F}_s + q_s q_t \overset{\curvearrowleft}{\mathbf{F}_s\Box}\mathbf{F}_t + q_t q_s \overset{\curvearrowleft}{\mathbf{F}_t\Box}\mathbf{F}_s + q_t q_t \overset{\curvearrowleft}{\mathbf{F}_t\Box}\mathbf{F}_t \\
& +q_s q_s \mathbf{F}_s \overset{\curvearrowright}{\Box\mathbf{F}_s} + q_s q_t \mathbf{F}_s \overset{\curvearrowright}{\Box\mathbf{F}_t} + q_t q_s \mathbf{F}_t \overset{\curvearrowright}{\Box\mathbf{F}_s} + q_t q_t \mathbf{F}_t \overset{\curvearrowright}{\Box\mathbf{F}_t} \\
= \;& \mu_0 \left[q_s \mathbf{V}_s q_s \mathbf{F}_s + q_s \mathbf{V}_s q_t \mathbf{F}_t + q_t \mathbf{V}_t q_s \mathbf{F}_s + q_t \mathbf{V}_t q_t \mathbf{F}_t \right. \\
& \left. -q_s \mathbf{F}_s q_s \mathbf{V}_s - q_s \mathbf{F}_s q_t \mathbf{V}_t - q_t \mathbf{F}_t q_s \mathbf{V}_s - q_t \mathbf{F}_t q_t \mathbf{V}_t\right]
\end{aligned}$$

The above displays all possible interactions.

On the right hand side write $\mathbf{V}_s\mathbf{F}_s = (\mathbf{V}_s\mathbf{F}_s - \mathbf{F}_s\mathbf{V})/2 + (\mathbf{V}_s\mathbf{F}_s + \mathbf{F}_s\mathbf{V}_s)/2$ and similarly for the other terms. Rearrange the right-hand side as follows:

$$\begin{aligned}
& \mu_0 \left[q_s \mathbf{V}_s q_s \mathbf{F}_s - q_s \mathbf{F}_s q_s \mathbf{V}_s + q_s \mathbf{V}_s q_t \mathbf{F}_t - q_s \mathbf{F}_t q_t \mathbf{V}_t \right. \\
& \left. +q_t \mathbf{V}_t q_s \mathbf{F}_s - q_t \mathbf{F}_t q_s \mathbf{V}_s + q_t \mathbf{V}_t q_t \mathbf{F}_t - q_t \mathbf{F}_t q_t \mathbf{V}_t\right]/2 \\
& +\mu_0 \left[-q_s q_s (\mathbf{F}_s\mathbf{V}_s - \mathbf{V}_s\mathbf{F}_s) - q_s \mathbf{F}_s q_t \mathbf{V}_t - q_t \mathbf{V}_t q_s \mathbf{F}_s \right. \\
& \left. -(q_t \mathbf{F}_t q_s \mathbf{V}_s - q_t \mathbf{V}_s q_s \mathbf{F}_t') - (q_t \mathbf{F}_t q_t \mathbf{V}_t - q_t \mathbf{V}_t q_t \mathbf{F}_t)\right] \\
= \;& -\mu_0 \left[q_s q_s (\mathbf{F}_s \mathbf{V}_s \overset{\text{IIa}}{-} \mathbf{V}_s \mathbf{F}_s) + q_t q_t (\mathbf{F}_t \mathbf{V}_t \overset{\text{IIb}}{-} \mathbf{V}_t \mathbf{F}_t) \right. \\
& \left. +(q_s q_t \mathbf{F}_s \mathbf{V}_t \overset{\text{Ia}}{-} q_t q_s \mathbf{V}_t \mathbf{F}_s) + (q_t q_s \mathbf{F}_t \mathbf{V}_s \overset{\text{Ib}}{-} q_t q_s \mathbf{V}_{\mathbf{s}} \mathbf{F}_t)\right]
\end{aligned} \tag{14.17}$$

Term Ia in Eq. (14.17) describes the force exerted by $q_s\mathbf{v}_s$ as it delivers a time rate of change of momentum into the field. Likewise term Ib is the force exerted by $q_t\mathbf{v}_t$ as it delivers a time rate of change of momentum into the field.

The mixed terms IIa and IIb, each involving both $q_s\mathbf{V}_\mathbf{s}$ and $q_t'\mathbf{V}_t$, describe the force delivered to the field, through the time rate of change of the Poynting momentum vector, as we shall show.

Consider Term Ia, call it $\mathbf{F}_{ts}$. The subscripts on $\mathbf{F}_{ts}$ represents the force of $q_s\mathbf{v}_s$ on $q_t\mathbf{v}_t$. Similarly $\mathbf{F}_{st}$ will represent the force $q_t\mathbf{v}_t$ on $q_s\mathbf{v}_s$. The force arises from the combination of $q_s\mathbf{v}_s$ and $q_t\mathbf{v}_t$ given by the mutual terms in the Poynting vector. We will call the force that arises from the fields the indirect force.

Recall that for all pairs of like multivectors $q_s q_t$,

$$\begin{aligned}
(q_s q_t + q_t q_s)/2 &= q_s \bullet q_t = \text{scalar} \qquad \mathbf{F}_s \bullet \mathbf{V}_t = (\mathbf{F}_s\mathbf{V}_t - \mathbf{V}_t\mathbf{F}_s)/2 = \text{vec} \\
(q_s q_t - q_t q_s)/2 &= q_s \wedge q_t = \text{biv} \qquad \mathbf{F}_s \wedge \mathbf{V}_t = (\mathbf{F}_s\mathbf{V}_t + \mathbf{V}_t\mathbf{F}_s)/2 = \text{triv}
\end{aligned}$$

$$\begin{aligned}\text{Term Ia} &= \mathbf{F}_{ts}^{\text{indir}} = \mu_0\left[q_s\mathbf{F}_s q_t\mathbf{V}_t - q_t\mathbf{V}_t q_s\mathbf{F}_s\right]/2\\ \mathbf{F}_{ts}^{\text{indir}} &= \mu_0\left[\left(\mathbf{F}_s\overset{\text{vec}}{\bullet}\mathbf{V}_t + \mathbf{F}_s\overset{\text{triv}}{\wedge}\mathbf{V}_t\right)q_sq_t - q_tq_s\left(\mathbf{V}_t\overset{\text{vec}}{\bullet}\mathbf{F}_s + \mathbf{V}_t\overset{\text{triv}}{\wedge}\mathbf{F}_s\right)\right]\end{aligned}$$

Rewrite, using $\mathbf{V}_t\bullet\mathbf{F}_s = -\mathbf{F}_s\bullet\mathbf{V}_t$ and $\mathbf{V}_t\wedge\mathbf{F}_s = \mathbf{F}_s\wedge\mathbf{V}_t$.
or explicitly

$$\begin{aligned}\text{Term Ia} = \mathbf{F}_{ts}^{\text{indir}} = \ &\mu_0\left[\left(\frac{(\mathbf{F}_s\mathbf{V}_t\overset{\text{vec}}{-}\mathbf{V}_t\mathbf{F}_s)}{2}\right)\overset{\text{scalar}}{(q_sq_t + q_tq_s)}/2\right.\\ &\left.+\left(\frac{(\mathbf{F}_s\mathbf{V}_t\overset{\text{triv}}{+}\mathbf{V}_t\mathbf{F}_s)}{2}\right)\left(q_sq_t\overset{\text{biv}}{-}q_tq_s\right)/2\right]\end{aligned} \tag{14.18}$$

The Terms Ia in Eq. (14.17), rearranged as in Eq. (14.18), are labeled $\mathbf{F}_{ts}^{\text{indir}}$ and identified as the electromagnetic force of $q_s\mathbf{v}_s$ on $q_t\mathbf{v}_t$, that is, the Poynting vector force. See Chapter 15 for a different approach to the above.

Term Ib in Eq. (14.17) gives the electromagnetic force on $q_t\mathbf{v}_t$ by reversing the label s and t in Eq. (14.18).

$$\begin{aligned}\text{Term Ib} \ = \ \mathbf{F}_{st}^{\text{indir}} = \mu_0&\left[\left(\frac{(\mathbf{F}_t\mathbf{V}_s\overset{\text{vec}}{-}\mathbf{V}_s\mathbf{F}_t)}{2}\right)\overset{\text{scalar}}{(q_tq_s + q_sq_t)}/2\right.\\ &\left.+\left(\frac{(\mathbf{F}_t\mathbf{V}_s\overset{\text{triv}}{+}\mathbf{V}_s\mathbf{F}_t)}{2}\right)\left(q_tq_s\overset{\text{biv}}{-}q_sq_t\right)/2\right]\end{aligned} \tag{14.19}$$

$\mathbf{F}_t$ in the above is:

$$\mathbf{F}_t = \frac{E_{tx}}{c}\mathbf{e}_0\mathbf{e}_1 + \frac{E_{ty}}{c}\mathbf{e}_0\mathbf{e}_2 + \frac{E_{tz}}{c}\mathbf{e}_0\mathbf{e}_3 + B_{tz}\mathbf{e}_1\mathbf{e}_2 + B_{ty}\mathbf{e}_3\mathbf{e}_1 + B_{tx}\mathbf{e}_2\mathbf{e}_3$$

14.3 Direct Force

The direct force of $q_s\mathbf{V}_s$ on $q_t\mathbf{V}_t$ is given by Eq. (14.18) when $\mathbf{F}_s$ is replaced by $\mathbf{B}_s$, that is, when

$$\begin{aligned}\mathbf{F}_{ts}^{\text{dir}} &\rightarrow \mu_0k\left(\mathbf{V}_s\mathbf{r}_{st}-\mathbf{r}_{st}\mathbf{V}_s\right)/2 = \mu_0k\left(\mathbf{V}_s\wedge\mathbf{r}_{st}\right) = \mathbf{B}_s\\ k &= 1/4\pi r^2\end{aligned}$$

For the direct interaction

$$\begin{aligned} \mathbf{V}_s &= (c\mathbf{e}_0 + v_{sx}\mathbf{e}_1 + v_{sy}\mathbf{e}_2 + v_{sz}\mathbf{e}_3) = (c\mathbf{e}_0 + \mathbf{v}_s) && (14.20)\\ \text{and}\quad \mathbf{V}_t &= (c\mathbf{e}_0 + v_{tx}\mathbf{e}_1 + v_{ty}\mathbf{e}_2 + v_{tz}\mathbf{e}_3) = (c\mathbf{e}_0 + \mathbf{v}_t) && (14.21)\end{aligned}$$

Explicitly the direct force of q_s on q_t is

$$\begin{aligned} \mathbf{F}_{ts}^{\text{dir}} &= \left[\frac{\left(\mathbf{B}_s\mathbf{V}_t \overset{\text{vec}}{-} \mathbf{V}_t\mathbf{B}_s\right)}{2}\frac{\left(q_sq_t \overset{\text{scalar}}{+} q_tq_s\right)}{2} + \frac{\left(\mathbf{B}_s\mathbf{V}_t \overset{\text{triv}}{+} \mathbf{V}_t\mathbf{B}_s\right)}{2}\frac{\left(q_sq_t \overset{\text{biv}}{-} q_tq_s\right)}{2}\right]\\ &= \left[\left(\mathbf{B}_s \overset{\text{vec}}{\bullet} \mathbf{V}_t\right)\left(q_sq_t \overset{\text{scalar}}{+} q_tq_s\right)/2 + \left(\mathbf{B}_s \overset{\text{triv}}{\wedge} \mathbf{V}_t\right)\left(q_sq_t \overset{\text{biv}}{-} q_tq_s\right)/2\right] \qquad (14.22)\end{aligned}$$

where $\mathbf{B}_s = \mu_0 k\,(\mathbf{V}_s \times \mathbf{r}_{st}),\ \ k = 1/4\pi r^3$.

The individual terms must be evaluated and contracted.

14.4 Values for $\mathbf{V}_t$ on $\mathbf{V}_s$ to Obtain the Indirect Force

The indirect force of $q_s\mathbf{V}_s$ on $q_t\mathbf{V}_t$ is given by Eq. (14.18)

For $\mathbf{V}_t$ and $\mathbf{V}_s$ substitute

$$\begin{aligned} \mathbf{e}_0\text{div}\frac{\mathbf{E}_t}{c} - \frac{1}{c^2}\frac{\partial\mathbf{E}_t}{c\partial t} + \text{curl}\mathbf{B}_t &= \mathbf{V}_t && (14.23)\\ -\mathbf{e}_0\text{div}\mathbf{B}_t + \frac{\partial\mathbf{B}_t}{\partial t} + \text{curl}\mathbf{E}_t &= 0 && (14.24)\\ \mathbf{e}_0\text{div}\frac{\mathbf{E}_s}{c} - \frac{1}{c^2}\frac{\partial\mathbf{E}_s}{\partial t} + \text{curl}\mathbf{B}_s &= \mathbf{V}_s && (14.25)\\ -\mathbf{e}_0\text{div}\mathbf{B}_s + \frac{\partial\mathbf{B}_t}{\partial t} + \text{curl}\mathbf{E}_s &= 0 && (14.26)\end{aligned}$$

$\mathbf{F}_s$ is given by

$$\mathbf{F}_s = \frac{E_{sx}}{c}\mathbf{e}_0\mathbf{e}_1 + \frac{E_{sy}}{c}\mathbf{e}_0\mathbf{e}_2 + \frac{E_{sz}}{c}\mathbf{e}_0\mathbf{e}_3 + B_{sz}\mathbf{e}_1\mathbf{e}_2 + B_{sy}\mathbf{e}_3\mathbf{e}_1 + B_{sx}\mathbf{e}_2\mathbf{e}_3 \qquad (14.27)$$

Replacing $\mathbf{B}_s$ by $\mathbf{F}_s$ in Eq. (14.22) we obtain

$$\mathbf{F}_{ts}^{\text{indir}} = (\mathbf{F}_s \overset{\text{vec}}{\bullet} \mathbf{V}_t)(q_sq_t \overset{\text{scalar}}{+} q_tq_s)/2 + (\mathbf{F}_s \overset{\text{triv}}{\wedge} \mathbf{V}_t)(q_sq_t \overset{\text{biv}}{-} q_tq_s)/2 \qquad (14.28)$$

Eq. (14.28) is

$$\mathbf{F}_{ts}^{\text{indir}} = (\mathbf{F}_s \overset{\text{vec}}{\bullet} \mathbf{V}_t)(q_s q_t \overset{\text{scalar}}{+} q_t q_s)/2 + (\mathbf{F}_s \overset{\text{triv}}{\wedge} \mathbf{V}_t)(q_s q_t \overset{\text{biv}}{-} q_t q_s)/2 \tag{14.29}$$

To emphasize that (14.29) with $\mathbf{F}_s$ replaced by $\mathbf{B}_t$ is applicable when calculating the direct force of $q_s\mathbf{V}_s$ on $q_t\mathbf{V}_t$, we write it again below after replacing $\mathbf{F}_s$ by $\mathbf{B}_s$, which is now Eq. (14.22).

$$\overset{\text{dir}}{\mathbf{F}}_{ts} = \left[(\mathbf{B}_s \overset{\text{vec}}{\bullet} \mathbf{V}_t)(q_s q_t \overset{\text{scalar}}{+} q_t q_s)/2 + (\mathbf{B}_s \overset{\text{triv}}{\wedge} \mathbf{V}_t)(q_s q_t \overset{\text{biv}}{-} q_t q_s)/2\right] \tag{14.30}$$

where $\mathbf{B}_s = \mu_0 k\left(\mathbf{V}_s \wedge \mathbf{r}_s\right)$ and the velocity terms$\mathbf{V}_t$ and $\mathbf{V}_s$ are given by (14.23) and (14.25) and $\mathbf{F}_s$ is defined by Eq. (14.27).

All terms must be evaluated, differentiated with respect to time, integrated over all space, and then contracted to obtain the final form of the force of $q_s\mathbf{V}_s$ on $q_t\mathbf{V}_t$. This order was followed in the initial orbital work.

In the following, we reverse the order of the above operations and perform the contraction first. The two sets of operations commute.

The contraction of a trivector times a bivector is always a vector. The combination $(q_s q_t + q_t q_s)/2$ is always a scalar. Any pair of multivectors of the form $(q_s q_t - q_t q_s)/2$ is always a bivector.

$$\begin{aligned} \text{Reminder} \qquad \mathbf{F}_s \bullet \mathbf{V}_t &= (\mathbf{F}_s\mathbf{V}_t - \mathbf{V}_t\mathbf{F}_s)/2 = \text{Vec} \\ \mathbf{F}_s \wedge \mathbf{V}_t &= (\mathbf{F}_s\mathbf{V}_t + \mathbf{V}_t\mathbf{F}_s)/2 = \text{Triv} \end{aligned}$$

Contraction: The recipe for contraction of the product of two non-commuting space-time multivectors $\mathbf{A}$ and B is to first express both in the same unit vectors. It can be either, but let us choose $\mathbf{A}$. After expressing $\mathbf{B}$ in the $\mathbf{A}$ unit-vectors, form

$$C(\mathbf{A}, \mathbf{B}) = \left(\mathbf{A}^{-1}\mathbf{B}^{-1} + \mathbf{B}^{-1}\mathbf{A}^{-1}\right)/2.$$

$\mathbf{A}^{-1}, \mathbf{B}^{-1}$ means reverse the unit vectors in both $\mathbf{A}$ and $\mathbf{B}$, substitute the reciprocal of each unit vector and carry out the multiplication in the $\mathbf{A}$ algebra, as indicated.

Re-arranging the terms in Eq. (14.29) as shown in (14.31), Term Ia comes from the "EM type" field and Term Ib is the contribution of the new field when q_s, q_t do not commute.

$$\mathbf{F}_{ts}^{\text{indir}} = \overbrace{\left(\mathbf{F}_s \overset{\text{vec}}{\bullet} \mathbf{V}_t\right)\left(q_s q_t \overset{\text{scalar}}{+} q_t q_s\right)}^{\textbf{Ia}}/2 + \overbrace{\left(\mathbf{F}_s \overset{\text{triv}}{\wedge} \mathbf{V}_t\right)\left(q_s q_t \overset{\text{biv}}{-} q_t q_s\right)}^{\textbf{Ib}}/2 \tag{14.31}$$

To repeat, in Eq.(14.31), Term Ia is the standard direct Lorentz force exerted by $q_s\mathbf{V}_s$ on $q_t\mathbf{V}_t$ when the field bivector (tensor) $\mathbf{F}_s \equiv \mathbf{B}_s$, is given by Eq.(14.32). When q_s, q_t do not commute one must add Ib to term Ia to obtain an additional force.

$$\mathbf{F}_s = \mathbf{B}_s = \mu_0 k\left(\mathbf{V}_s \wedge \mathbf{r}\right) \qquad k = 1/4\pi r^3 \tag{14.32}$$

The currents $\mathbf{V}_t$ and $\mathbf{V}_s$, are related to the $\mathbf{E}_t$, $\mathbf{B}_t$ and $\mathbf{E}_s$, $\mathbf{B}_s$ fields by Eq. (14.23)and (14.25). Eqs.(14.23) and (14.25) give the first set of Maxwell's equations for each of the sources.

Eqs. (14.24) and (14.26) provide a relationship between field quantities $\mathbf{E}$ and $\mathbf{B}$, and constitute the second set of Maxwell's equations.

Interchanging s and t in the above equations and replacing $\mathbf{r}_{st}$ by $\mathbf{r}_{ts}$ where $\mathbf{r}_{ts} = -\mathbf{r}_{st}$ gives force of the field on $q_s\mathbf{V}_s$.

Note again that the first line of Eq. (14.31), namely

$$\mathbf{F}_{ts} = \frac{\left(\mathbf{F}_s\mathbf{V}_t \overset{\text{vec}}{-} \mathbf{V}_t\mathbf{F}_s\right)}{2}\left(q_s q_t \overset{\text{scaler}}{+} q_t q_s\right)/2 + \frac{\left(\mathbf{F}_s\mathbf{V}_t \overset{\text{triv}}{+} \mathbf{V}_t\mathbf{F}_s\right)}{2}\left(q_s q_t \overset{\text{biv}}{-} q_t q_s\right)/2 \tag{14.33}$$

may be used to obtain the direct force of $q_s\mathbf{v}_s$ on $q_t\mathbf{v}_t$ by replacing $\mathbf{V}_t$ by the physical four-current

$$\mathbf{V}_t = \left(c\mathbf{e}_0 + v_{tx}\mathbf{e}_1 + v_{ty}\mathbf{e}_2 + v_{tz}\mathbf{e}_3\right) = \left(c\mathbf{e}_0 + \mathbf{v}_t\right)$$

and replacing $\mathbf{F}_s$ by $\mathbf{B}_s$, that is,

$$\mathbf{F}_s \equiv \mathbf{B}_s = \mu_0 k\left(\mathbf{V}_s \wedge \mathbf{r}_{st}\right) \qquad k = 1/4\pi r^3 \tag{14.34}$$

where $\mathbf{V}_s$ is given by Eq. (14.20) .

To obtain the indirect, that is the electromagnetic, force associated with time rate of change of momentum imparted to the field, we need to evaluate the vector

$$\mathbf{F}_{ts} = \left(\mathbf{F}_s \overset{\text{vec}}{\bullet} \mathbf{V}_t\right)\left(q_s q_t \overset{\text{scalar}}{+} q_t q_s\right)/2 \tag{14.35}$$

with $\mathbf{V}_t$ given by Eq.(14.23) and substitute the field bivector, $\mathbf{F}_s$, also called the field tensor, expressed in terms of the field components, namely

$$\mathbf{F}_s = \frac{E_{sx}}{c}\mathbf{e}_0\mathbf{e}_1 + \frac{E_{sy}}{c}\mathbf{e}_0\mathbf{e}_2 + \frac{E_{sz}}{c}\mathbf{e}_0\mathbf{e}_3 + B_{sz}\mathbf{e}_1\mathbf{e}_2 + B_{sy}\mathbf{e}_3\mathbf{e}_1 + B_{sx}\mathbf{e}_2\mathbf{e}_3 \tag{14.36}$$

Adding these 2 contributions, we obtain the time rate of change of the momentum imparted to the field associated with $q_s\mathbf{V}_s$ acting on $q_t\mathbf{V}_t$, both directly and indirectly.

Interchange labels s and t to obtain the corresponding expression for the time rate of change of momentum associated with $q_t\mathbf{V}_t$ acting on $q_s\mathbf{V}_s$ and replace $\mathbf{r}_{st}$ by $-\mathbf{r}_{ts}$.

The second group of terms in Eq. (14.31) is the trivector-biv.

$$\left(\mathbf{F}_s \overset{\text{triv}}{\wedge} \mathbf{V}_t\right)\left(q_s q_t \overset{\text{biv}}{-} q_t q_s\right)/2 \tag{14.37}$$

All equations must now be "contracted" to obtain the final expression. When the "charges" q_s, q_t are scalars, the terms $q_t q_s - q_s q_t$ vanish. When the "charges" are four-momenta P_s, P_t, the case to which we specialize for gravity, $P_s P_t - P_t P_s$ is a bivector, which contracted with a trivector, is a vector and therefore has the required form for a four-vector force. $P_s P_t + P_t P_s$ is a scalar and contraction has no special effect. Note that we have not used heavy case letters for the four-momentum P_s and P_t. These are space-time vectors, however, and simply replace q_s and q_t which designate any pair of space-time vectors. Important: The combination $q_s q_t - q_t q_s$ is a bivector for all multivectors.

14.5 Detailed Evaluation of the Vector Direct Force

$$\left(\mathbf{B}_s \overset{\text{triv}}{\wedge} \mathbf{V}_t\right)\left(q_s q_t \overset{\text{biv}}{-} q_t q_s\right)/2 = \frac{\left(\mathbf{B}_s\mathbf{V}_t \overset{\text{triv}}{+} \mathbf{V}_t\mathbf{B}_s\right)}{2}\left(q_s q_t \overset{\text{biv}}{-} q_t q_s\right)/2 \tag{14.38}$$

$$\begin{aligned}\mathbf{B}_s \wedge \mathbf{V}_t &= \mu_0 k\left\{\left[(\mathbf{v}_s \times \mathbf{r}_{st})_z - (\mathbf{v}_t \times \mathbf{r}_{st})_z\right]\mathbf{e}_0\mathbf{e}_1\mathbf{e}_2\right.\\ &\quad + \left[c\,(\mathbf{v}_s \times \mathbf{r}_{st})_y - (\mathbf{v}_t \times \mathbf{r}_{st})_y\right]\mathbf{e}_0\mathbf{e}_3\mathbf{e}_1\\ &\quad \left.+ \left[c\,(\mathbf{v}_s \times \mathbf{r}_{st})_x - (\mathbf{v}_t \times \mathbf{r}_{st})_x\right]\mathbf{e}_0\mathbf{e}_2\mathbf{e}_3 + \left[\mathbf{v}_t \cdot (\mathbf{v}_s \times \mathbf{r}_{st})\right]\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\right\}\end{aligned} \tag{14.39}$$

Multiply by $-\mathbf{e}_5\mathbf{e}_5 = 1$

$$\begin{aligned}\mathbf{B}_s \overset{\text{triv}}{\wedge} \mathbf{V}_t &= -\mathbf{e}_5\mathbf{e}_5\mu_0 k\left\{c\left[(\mathbf{v}_s \times \mathbf{r}_{st})_z - (\mathbf{v}_t \times \mathbf{r}_{st})_z\right]\mathbf{e}_0\mathbf{e}_1\mathbf{e}_2\right.\\ &\quad + \left[c\,(\mathbf{v}_s \times \mathbf{r}_{st})_y - (\mathbf{v}_t \times \mathbf{r}_{st})_y\right]\mathbf{e}_0\mathbf{e}_3\mathbf{e}_1\\ &\quad \left.+ \left[c\,(\mathbf{v}_s \times \mathbf{r}_{st})_x - (\mathbf{v}_t \times \mathbf{r}_{st})_x\right]\mathbf{e}_0\mathbf{e}_2\mathbf{e}_3 + \left[\mathbf{v}_t \cdot (\mathbf{v}_s \times \mathbf{r}_{st})\right]\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\right\}\\ &= -\mathbf{e}_5\mu_0 k\left\{-\left[(\mathbf{v}_s \times \mathbf{r}_{st})_z - (\mathbf{v}_t \times \mathbf{r}_{st})_z\right]\mathbf{e}_3 - \left[c\,(\mathbf{v}_s \times \mathbf{r}_{st})_y - (\mathbf{v}_t \times \mathbf{r}_{st})_y\right]\mathbf{e}_2\right.\\ &\quad \left.- \left[c\,(\mathbf{v}_s \times \mathbf{r}_{st})_x - (\mathbf{v}_t \times \mathbf{r}_{st})_x\right]\mathbf{e}_3 - \left[\mathbf{v}_t \cdot (\mathbf{v}_s \times \mathbf{r}_{st})\right]\mathbf{e}_0\right\}\\ &= \mathbf{e}_5\mu_0 k\left\{\mathbf{e}_0\left[\mathbf{v}_t \cdot (\mathbf{v}_s \times \mathbf{r}_{st})\right] + \mathbf{e}_1\left[c\,(\mathbf{v}_s \times \mathbf{r}_{st})_x - (\mathbf{v}_t \times \mathbf{r}_{st})_x\right]\right.\\ &\quad \left.+\mathbf{e}_2\left[c\,(\mathbf{v}_s \times \mathbf{r}_{st})_y - (\mathbf{v}_t \times \mathbf{r}_{st})_y\right] + \mathbf{e}_3\left[(\mathbf{v}_s \times \mathbf{r}_{st})_z - (\mathbf{v}_t \times \mathbf{r}_{st})_z\right]\right\}\end{aligned}$$

which may be written,

$$(\mathbf{B}_s \wedge \mathbf{V}_t)(q_s q_t - q_t q_s)/2 = \mu_0 k \mathbf{e}_5 \{\mathbf{e}_0 [\mathbf{v}_t \cdot (\mathbf{v}_s \times \mathbf{r}_{st})] + c[(\mathbf{v}_s \times \mathbf{r}_{st}) - (\mathbf{v}_t \times \mathbf{r}_{st})]\}(q_s q_t - q_t q_s)/2 \quad (14.40)$$

or, since $\mathbf{v}_s \cdot (\mathbf{r}_{st} \times \mathbf{v}_t) \rightarrow \mathbf{r}_{st} \cdot (\mathbf{v}_t \times \mathbf{v}_s)$

$$(\mathbf{B}_s \wedge \mathbf{V}_t)(q_s q_t - q_t q_s)/2 = \mu_0 k \mathbf{e}_5 \{\mathbf{e}_0 [\mathbf{r}_{st} \cdot (\mathbf{v}_t \times \mathbf{v}_s)] + c[(\mathbf{v}_s \times \mathbf{r}_{st}) - (\mathbf{v}_t \times \mathbf{r}_{st})]\}(q_s q_t - q_t q_s)/2 \quad (14.41)$$

If s and t are interchanged, Eq.(14.41) becomes

$$(\mathbf{B}_t \wedge \mathbf{V}_s)(q_t q_s - q_s q_t)/2 = \mu_0 k \mathbf{e}_5 \{\mathbf{e}_0 [\mathbf{r}_{ts} \cdot (\mathbf{v}_s \times \mathbf{v}_t)] + c[(\mathbf{v}_t \times \mathbf{r}_{ts}) - (\mathbf{v}_s \times \mathbf{r}_{ts})]\}(q_t q_s - q_s q_t)/2 \quad (14.42)$$

Since $\mathbf{r}_{ts} = -\mathbf{r}_{st}$, and $(\mathbf{v}_s \times \mathbf{v}_t) = -(\mathbf{v}_t \times \mathbf{v}_s)$, Eq.(14.41) may be written

$$(\mathbf{B}_t \wedge \mathbf{V}_s)(q_t q_s - q_s q_t)/2 = \mu_0 k c \mathbf{e}_5 \{\mathbf{e}_0 [\mathbf{r}_{st} \cdot (\mathbf{v}_t \times \mathbf{v}_s)] + c[(\mathbf{v}_s \times \mathbf{r}_{st}) - (\mathbf{v}_t \times \mathbf{r}_{st})]\}(q_t q_s - q_s q_t)/2 \quad (14.43)$$

Thus Eq.(14.41), when written with the changes $t \rightarrow s$ and $s \rightarrow t$ and $\mathbf{r}_{ts} = -\mathbf{r}_{st}$, becomes Eq.(14.43) which is the same as Eq.(14.41) and does not change sign. Note that

$$(q_s q_t - q_t q_s)/2 = -(q_t q_s - q_s q_t)/2$$

In the case of gravity $q_t = P_t$ and $q_s = P_s$ where P_t and P_s are then gravitational charges. Note that $P_s P_t - P_t P_s$ changes sign when s and t are interchanged. If the non-sign change of Eq.(14.43) holds after contraction, then Newton's third law will hold for the direct force of $q_s \mathbf{V}_s$ on $q_t \mathbf{V}_t$ since the direct force of $q_t \mathbf{V}_t$ on $q_s \mathbf{V}_s$ is opposite in sign. We can check this out by referring to the Tables on Contraction and see whether a trivector multiplied by a bivector and then contracted changes sign when s and t are interchanged. It does. The same applies if the coefficient of $\mathbf{e}_0$ is not present.

We have just verified that the coefficients of the trivector, Eq.(14.41) do not change sign when s and t are interchanged, including the substitution $\mathbf{r}_{ts} = -\mathbf{r}_{st}$. Referring to contraction Table IV, item 7, we see that the coefficient s in the space time bivector all reverse sign when s and t are reversed in the bivector coefficients. Therefore, in the final result, a victor tabulated therein each of the b^{ij} coefficients experience a change of sign when the quantities defining the b^{ij} coefficients undergo a reversal in s and t. Therefore, the net result is that the sign of the force of $q_s \mathbf{V}_s$ on $q_t \mathbf{V}_t$ is the negative of the force of $q_t \mathbf{V}_t$ on $q_s \mathbf{V}_s$ and Newton's third law is satisfied.

Thus, Eq.(14.41) does not change sign when re-written with $t \to s$ and $s \to t$ and $\mathbf{r}_{ts} = -\mathbf{r}_{st}$. However, to obtain the force associated with Eq.(14.40), it must be multiplied by $P_t \bullet P_s$ and contracted. $P_t \bullet P_s$ changes sign when $\mathbf{v}_t$, $\mathbf{v}_s$ are interchanged. Contracting (14.42) and replacing P_s, P_t where

$$\begin{aligned} P_s &= m_s \mathbf{V}_s = \mathbf{e}_0 c m_s + p_s = \mathbf{e}_0 c m_s + m_s \mathbf{v}_s \\ P_t &= m_t \mathbf{V}_t = \mathbf{e}_0 c m_t + p_t \end{aligned}$$

Omitting the $\mathbf{e}_0$ term and contracting, we obtain

$$\begin{aligned} \mathbf{F}_{ts} &= \mu_0 k \left\{ c^2 \left[\mathbf{v}_s \left(\mathbf{v}_s \cdot \mathbf{r}_{st} \right) - \mathbf{r}_{st} \mathbf{v}_s^2 + \mathbf{v}_s \times \left(\mathbf{r}_{st} \times \mathbf{v}_t \right) \right.\right. \\ &\quad \left. + \mathbf{v}_t \left(\mathbf{v}_t \cdot \mathbf{r}_{st} \right) + \mathbf{r}_{st} \left(\mathbf{v}_s \cdot \mathbf{v}_t \right) + \mathbf{v}_t \times \left(\mathbf{r}_{st} \times \mathbf{v}_s \right) \right] \\ &\quad \left. + \left[\mathbf{v}_t \cdot \left(\mathbf{v}_s \times \mathbf{r}_{st} \right) \right] \left(\mathbf{v}_t \times \mathbf{v}_s \right) \right\} \end{aligned} \tag{14.44}$$

The first three terms also occur in the coefficient of $\mathbf{e}_0$.

Interchange s and t above and replace $\mathbf{r}_{ts}$ by $-\mathbf{r}_{st}$ to obtain

$$\begin{aligned} \mathbf{F}_{st} &= \mu_0 k \left\{ c^2 \left[\mathbf{v}_t \left(\mathbf{v}_t \cdot \mathbf{r}_{ts} \right) - \mathbf{r}_{ts} \mathbf{v}_t^2 + \mathbf{v}_t \times \left(\mathbf{r}_{ts} \times \mathbf{v}_s \right) \right.\right. \\ &\quad \left. + \mathbf{v}_s \left(\mathbf{v}_s \cdot \mathbf{r}_{ts} \right) + \mathbf{r}_{ts} \left(\mathbf{v}_t \cdot \mathbf{v}_s \right) + \mathbf{v}_s \times \left(\mathbf{r}_{ts} \times \mathbf{v}_t \right) \right] \\ &\quad \left. + \mathbf{v}_s \cdot \left(\mathbf{v}_t \times \mathbf{r}_{ts} \right) \left(\mathbf{v}_s \times \mathbf{v}_t \right) \right\} \end{aligned} \tag{14.45}$$

14.6 Evaluation of the Indirect Force (cont.)

$$\begin{aligned} q_s q_s \left(\mathbf{F}_s \overset{\curvearrowright}{\square} \mathbf{F}_s + \mathbf{F}_s \overset{\curvearrowleft}{\square} \mathbf{F}_s \right) &= q_s q_s \left(\mathbf{F}_s \mathbf{V}_s - \mathbf{V}_s \mathbf{F}_s \right) / 2 \\ &= q_s q_s \left(\mathbf{F}_s \bullet \mathbf{V}_s \right) \end{aligned} \tag{14.46}$$

To evaluate $(\mathbf{F}_s \bullet \mathbf{V}_s)$, use

$$\mathbf{V}_s = \mathbf{e}_0 \mathrm{div} \frac{\mathbf{E}_s}{c} - \frac{1}{c^2} \frac{\partial \mathbf{E}_s}{\partial t} + \mathrm{curl} \mathbf{B}_s \tag{14.47}$$

and

$$\mathbf{F}_s = \mathbf{e}_0 \mathbf{e}_1 \frac{\mathbf{E}_{sx}}{c} + \mathbf{e}_0 \mathbf{e}_2 \frac{E_{sy}}{c} + \mathbf{e}_0 \mathbf{e}_3 \frac{E_{sz}}{c} + \mathbf{e}_1 \mathbf{e}_2 B_{sz} + \mathbf{e}_3 \mathbf{e}_1 B_{sy} + \mathbf{e}_2 \mathbf{e}_3 B_{sx}$$

$$\begin{aligned} \mathbf{F}_s \bullet \mathbf{V}_s &= \left(\mathbf{F}_s \mathbf{V}_s - \mathbf{V}_s \mathbf{F}_s \right) / 2 \qquad \mathrm{curl} \mathbf{B}_s = \frac{1}{c^2} \frac{\partial \mathbf{E}_s}{\partial t} \\ &= \mathbf{e}_0 \left[-\frac{1}{c^3} \frac{\partial}{\partial t} \left(\frac{\mathbf{E}_s^2}{2} \right) + \frac{\mathbf{E}_s}{c} \cdot \mathrm{curl} \mathbf{B}_s \right] \\ &\quad - \frac{1}{c^2} \left(\frac{\partial \mathbf{E}_s}{\partial t} \right) \times \mathbf{B}_s + \mathbf{B}_s \times \ \mathrm{curl} \mathbf{B}_s - \left(\mathrm{div} \frac{\mathbf{E}_s}{c} \right) \mathbf{E}_s \end{aligned} \tag{14.48}$$

$$-\mathbf{e}_0 \mathrm{div} \mathbf{B}_s + \frac{\partial \mathbf{B}_s}{c\partial t} + \mathrm{curl} \frac{\mathbf{E}_s}{c} = 0 \tag{14.49}$$

Eq.(14.49) is the second set of Maxwell's equations.

Now evaluate

$$\mathbf{e}_0 \left[-\frac{1}{c} \mathbf{B}_s \cdot \mathrm{curl}\, \mathbf{E}_s + \frac{1}{c} \frac{\partial}{\partial t} \frac{\mathbf{B}_s^2}{2} - \frac{1}{c^2} \left(\mathbf{E}_s \times \mathrm{curl} \mathbf{E}_s \right) - \frac{\mathbf{E}_s}{c} \times \left(\frac{\partial \mathbf{B}_s}{c\partial t} \right) + \left(\mathrm{div} \mathbf{B}_s \right) \mathbf{B}_s \right] = 0 \tag{14.50}$$

Combine Eqs.(14.48) and (14.50) to obtain

$$\begin{aligned} \mathbf{F}_s \bullet \mathbf{V}_s & \\ = \ & \mathbf{e}_0 \left[-\frac{1}{c} \frac{\partial}{\partial t} \left(\frac{\mathbf{E}_s^2}{2c^2} \right) + \frac{\mathbf{E}_s}{c} \cdot \mathrm{curl}\, \mathbf{B}_s - \underline{\frac{\partial}{c\partial t} \left(\frac{\mathbf{B}_s^2}{2} \right)} + \underline{\mathbf{B}_s \cdot \mathrm{curl}\, \frac{\mathbf{E}_s}{c}} \right] \\ & -\frac{1}{c^2} \frac{\partial \mathbf{E}_s}{\partial t} \times \mathbf{B}_s - \underline{\frac{\mathbf{E}_s}{c^2} \times \frac{\partial \mathbf{B}_s}{\partial t}} - \mathbf{B}_s \times \mathrm{curl}\, \mathbf{B}_s - \underline{\frac{1}{c^2} \mathbf{E}_s \times \mathrm{curl}\, \mathbf{E}_s} \\ & + \left(\mathrm{div}\, \frac{\mathbf{E}_s}{c} \right) \frac{\mathbf{E}_s}{c} + \underline{\mathbf{B}_s \mathrm{div}\, \mathbf{B}_s} \end{aligned} \tag{14.51}$$

which result is also given in Jackson, p. 238.

Re-writing (14.51)

$$\begin{aligned} \mathbf{F}_s \bullet \mathbf{V}_s = \ & \mathbf{e}_0 \left[-\frac{\partial}{\partial t} \frac{1}{2} \left(\frac{\mathbf{E}_s^2}{c^2} + \underline{\mathbf{B}_s^2} \right) + \frac{\mathbf{E}_s}{c} \cdot \mathrm{curl} \mathbf{B}_s + \underline{\mathbf{B}_s \cdot \mathrm{curl} \frac{\mathbf{E}_s}{c}} \right] \\ & -\frac{\partial}{\partial t} \underline{\underline{\frac{(\mathbf{E}_s \times \mathbf{B}_s)}{c^2}}} - \mathbf{B}_s \times \mathrm{curl} \mathbf{B}_s - \frac{1}{c^2} \underline{\mathbf{E}_s \times \mathrm{curl} \mathbf{E}_s} \\ & + \frac{\mathbf{E}_s}{c} \mathrm{div} \frac{\mathbf{E}_s}{c} + \underline{\mathbf{B}_s \mathrm{div} \mathbf{B}_s} \end{aligned} \tag{14.52}$$

The double underline means that a contribution comes from an underlined term and a non-underlined term. Eq. (14.52) is the force density in the field. The term $\frac{1}{2} \left(\frac{\mathbf{E}_s^2}{c^2} + \mathbf{B}_s^2 \right)$ is the energy density in the field. The term $\left(\frac{\mathbf{E}_s}{c} \times \mathbf{B}_s \right)$ is the momentum density in the field.

We now evaluate

$$\mathbf{F}_s \overset{\mathrm{vec}}{\bullet} \mathbf{V}_t = \left(\mathbf{F}_s \mathbf{V}_t - \mathbf{V}_t \mathbf{F}_s \right) / 2$$

which is the force flow into the field generated by the "field" currents $q_s \mathbf{V}_s$ and $q_t \mathbf{V}_t$ and where $\mathbf{F}_s$ is

$$\mathbf{F}_s = \mathbf{e}_0 \mathbf{e}_1 \frac{E_{sx}}{c} + \mathbf{e}_0 \mathbf{e}_2 \frac{E_{sy}}{c} + \mathbf{e}_0 \mathbf{e}_3 \frac{E_{sz}}{c} + \mathbf{e}_1 \mathbf{e}_2 B_{sz} + \mathbf{e}_3 \mathbf{e}_1 B_{sy} + \mathbf{e}_2 \mathbf{e}_3 B_{sx} \tag{14.53}$$

Using $\mathbf{V}_t = \mathbf{e}_0 \mathrm{div}\frac{\mathbf{E}_t}{c} - \frac{1}{c^2}\frac{\partial \mathbf{E}_t}{\partial t} + \mathrm{curl}\mathbf{B}_t$ gives
$\left(\mathbf{F}_s\mathbf{V}_t - \mathbf{V}_t\mathbf{F}_s\right)/2 = \mathbf{F}_s \bullet \mathbf{V}_t$

$$
\begin{aligned}
= \ & \mathbf{e}_0 \left[\left(-\frac{1}{c^2}\frac{\partial E_{tx}}{\partial t} + (\mathrm{curl}\mathbf{B}_t)_x\right)\frac{E_{sx}}{c} + \left(-\frac{1}{c^2}\frac{\partial E_{ty}}{\partial t} + (\mathrm{curl}\mathbf{B}_t)_y\right)\frac{E_{sy}}{c}\right. \\
& \left. + \left(-\frac{1}{c^2}\frac{\partial E_{tz}}{\partial t} + (\mathrm{curl}\mathbf{B}_t)_z\right)\frac{E_{sz}}{c}\right] \\
& + \mathbf{e}_1 \left[\left(-\frac{1}{c^3}\frac{\partial E_{ty}}{\partial t} + (\mathrm{curl}\mathbf{B}_t)_y\right)B_{sz} - \left(-\frac{1}{c^2}\frac{\partial E_{tz}}{\partial t} + (\mathrm{curl}\mathbf{B}_t)_z\right)B_{sy}\right. \\
& \left. + \mathrm{div}\frac{\mathbf{E}_t}{c}\frac{E_{sx}}{c}\right] + \mathbf{e}_2\left[\left(-\frac{1}{c^2}\frac{\partial E_{tz}}{\partial t} + (\mathrm{curl}\mathbf{B}_t)_z\right)B_{sx}\right. \\
& \left. - \left(-\frac{1}{c^2}\frac{\partial E_{tx}}{\partial t} + (\mathrm{curl}\mathbf{B}_t)_x\right)B_{sz} + \left(\mathrm{div}\frac{\mathbf{E}_t}{c}\right)\frac{E_{sy}}{c}\right] \\
& + \mathbf{e}_3\left[\left(-\frac{1}{c^2}\frac{\partial E_{tx}}{\partial t} + (\mathrm{curl}\mathbf{B}_t)_x\right)B_{sy} - \left(-\frac{1}{c^2}\frac{\partial E_{ty}}{\partial t} + (\mathrm{curl}\mathbf{B}_t)_y\right)B_{sx}\right. \\
& \left.\left. + \left(\mathrm{div}\frac{\mathbf{E}_t}{c}\right)\frac{E_{sz}}{c}\right]\right\}
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{F}_s \bullet \mathbf{V}_t = \ & -\mathbf{e}_0\left(-\frac{\partial \mathbf{E}_t}{c^2\partial t} + \mathrm{curl}\mathbf{B}_t\right)\cdot\frac{\mathbf{E}_s}{c} \\
& + \left(-\frac{1}{c^2}\frac{\partial \mathbf{E}_t}{\partial t} + \mathrm{curl}\mathbf{B}_t\right)\times\mathbf{B}_s + \left(\mathrm{div}\frac{\mathbf{E}_t}{c}\right)\frac{\mathbf{E}_s}{c}
\end{aligned}
$$

14.7 Summary

$$\begin{aligned}\mathbf{F}_s \bullet \mathbf{V}_t &= (\mathbf{F}_s\mathbf{V}_t - \mathbf{V}_t\mathbf{F}_s)/2 = \left\{\mathbf{e}_0\left[\left(-\frac{1}{c^2}\frac{\partial \mathbf{E}_t}{\partial t}\right)\cdot\frac{\mathbf{E}_s}{c} + \frac{\mathbf{E}_s}{c}\cdot \mathrm{curl}\mathbf{B}_t\right]\right.\\ &\left.-\frac{1}{c^2}\frac{\partial \mathbf{E}_t}{\partial t}\times\mathbf{B}_s + \mathbf{B}_s\times\mathrm{curl}\mathbf{B}_t + \left(\mathrm{div}\frac{\mathbf{E}_t}{c}\right)\frac{\mathbf{E}_s}{c}\right\} \qquad (14.54)\\ \mathbf{F}_t \bullet \mathbf{V}_s &= (\mathbf{F}_t\mathbf{V}_s - \mathbf{V}_s\mathbf{F}_t)/2 = \left\{\mathbf{e}_0\left[\left(-\frac{1}{c^2}\frac{\partial \mathbf{E}_s}{\partial t}\right)\cdot\frac{\mathbf{E}_t}{c} + \frac{\mathbf{E}_t}{c}\cdot \mathrm{curl}\mathbf{B}_s\right]\right.\\ &\left.-\frac{1}{c^2}\frac{\partial \mathbf{E}_s}{\partial t}\times\mathbf{B}_t + \mathbf{B}_t\times\mathrm{curl}\mathbf{B}_s + \left(\mathrm{div}\frac{\mathbf{E}_s}{c}\right)\frac{\mathbf{E}_t}{c}\right\} \qquad (14.55)\\ \mathbf{F}_s \bullet \mathbf{V}_s &= (\mathbf{F}_s\mathbf{V}_s - \mathbf{V}_s\mathbf{F}_s)/2 = \left\{\mathbf{e}_0\left[\left(-\frac{1}{c^2}\frac{\partial \mathbf{E}_s}{\partial t}\right)\cdot\frac{\mathbf{E}_s}{c} + \frac{\mathbf{E}_s}{c}\cdot \mathrm{curl}\mathbf{B}_s\right]\right.\\ &\left.-\frac{1}{c^2}\frac{\partial \mathbf{E}_s}{\partial t}\times\mathbf{B}_s + \mathbf{B}_s\times\mathrm{curl}\mathbf{B}_s + \left(\mathrm{div}\frac{\mathbf{E}_s}{c}\right)\frac{\mathbf{E}_s}{c}\right\} \qquad (14.56)\\ \mathbf{F}_t \bullet \mathbf{V}_t &= (\mathbf{F}_t\mathbf{V}_t - \mathbf{V}_t\mathbf{F}_t)/2 = \mathbf{e}_0\left\{\left[\left(-\frac{1}{c^2}\frac{\partial \mathbf{E}_t}{\partial t}\right)\cdot\frac{\mathbf{E}_t}{c} + \frac{\mathbf{E}_t}{c}\cdot \mathrm{curl}\mathbf{B}_t\right]\right.\\ &\left.-\frac{1}{c^2}\frac{\partial \mathbf{E}_t}{\partial t}\times\mathbf{B}_t + \mathbf{B}_t\times\mathrm{curl}\mathbf{B}_t + \left(\mathrm{div}\frac{\mathbf{E}_t}{c}\right)\frac{\mathbf{E}_t}{c}\right\} \qquad (14.57)\end{aligned}$$

The above terms are not zero, therefore, they describe a field.

Thus

$$\begin{aligned}&\frac{(\mathbf{F}_s\mathbf{V}_t - \mathbf{V}_t\mathbf{F}_s)}{2}(q_s q_t + q_t q_s)/2 \equiv \mathbf{F}_s \bullet \mathbf{V}_t (q_s q_t + q_t q_s)\\ &= \left\{\mathbf{e}_0\left[\left(-\frac{1}{c^2}\frac{\partial \mathbf{E}_t}{\partial t}\right)\cdot\frac{\mathbf{E}_s}{c} + \frac{\mathbf{E}_s}{c}\cdot\mathrm{curl}\mathbf{B}_t\right]\right.\\ &\left.-\frac{1}{c^2}\frac{\partial \mathbf{E}_t}{\partial t}\times\mathbf{B}_s + \mathbf{B}_s\times\mathrm{curl}\mathbf{B}_t + \left(\mathrm{div}\frac{\mathbf{E}_t}{c}\right)\frac{\mathbf{E}_s}{c}\right\}(q_s q_t + q_t q_s)/2\end{aligned}$$

For gravity

$$\begin{aligned}q_t &= P_t = \gamma_{v'_t}(m_t\mathbf{e}_0 c + \mathbf{p}_t) = \gamma_{v'_t} m_t (e_0 c + \mathbf{v}'_t)\\ P_t P_t &= \gamma^2_{v'_t} m_t^2\left(-c^2 + v'^2_t\right) \qquad p_t = m_t v_t\end{aligned}$$

Evaluate

$$\begin{aligned}
\mathbf{F}_s \overset{\text{triv}}{\wedge} \mathbf{V}_t &= \left(\mathbf{F}_s\mathbf{V}_t \overset{\text{triv}}{+} \mathbf{V}_t\mathbf{F}_s\right)/2 \\
&= \left(cB_s^{12} - \upsilon_{tx}B_s^{02} + \upsilon_{ty}B_s^{01}\right)\mathbf{e}_0\mathbf{e}_1\mathbf{e}_2 + \left(cB_s^{31} + \upsilon_{tx}B_s^{02} - \upsilon_{tz}B_s^{01}\right)\mathbf{e}_0\mathbf{e}_3\mathbf{e}_1 \\
&\quad + \left(cB_s^{23} - \upsilon_{ty}B_s^{03} + \upsilon_{tz}B_s^{02}\right)\mathbf{e}_0\mathbf{e}_2\mathbf{e}_3 \\
&\quad + \left(\upsilon_{tx}B_s^{23} + \upsilon_{ty}B_s^{31} + \upsilon_{tz}B_s^{12}\right)\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 \qquad (14.58) \\
\mathbf{V}_t &= \mathbf{e}_0\mathrm{div}\frac{\mathbf{E}_t}{c} - \frac{1}{c^2}\frac{\partial \mathbf{E}_t}{\partial t} + \mathrm{curl}\mathbf{B}_t \qquad (14.59) \\
\mathbf{F}_s &= \mathbf{e}_0\mathbf{e}_1\frac{E_{sx}}{c} + \mathbf{e}_0\mathbf{e}_2\frac{E_{sy}}{c} + \mathbf{e}_0\mathbf{e}_3\frac{E_{sz}}{c} + \mathbf{e}_1\mathbf{e}_2 B_{sz} + \mathbf{e}_3\mathbf{e}_1 B_{sy} + \mathbf{e}_2\mathbf{e}_3 B_{sx} \qquad (14.60)
\end{aligned}$$

$$\begin{aligned}
&\left(\mathbf{F}_s \overset{\text{triv}}{\wedge} \mathbf{V}_t\right)\left(q_s q_t \overset{\text{biv}}{-} q_t q_s\right)/2 \\
&= \left[\left(\mathbf{F}_s\mathbf{V}_t + \mathbf{V}_t\mathbf{F}_s\right)/2\right]\left(q_s q_t - q_t q_s\right)/2 \\
&= \Bigg\{\mathbf{e}_0\mathbf{e}_1\mathbf{e}_2\left[-\left(\mathrm{div}\frac{\mathbf{E}_t}{c}\right)B_{sz} - \frac{1}{c^2}\frac{\partial E_{tx}}{\partial t}E_{sy} + \left(\mathrm{curl}\mathbf{B}_t\right)_x\frac{E_{sy}}{c}\right. \\
&\quad \left. + \frac{1}{c^2}\frac{\partial E_{ty}}{\partial t}\frac{E_{sx}}{c} - \left(\mathrm{curl}\mathbf{B}_t\right)_y E_{sx}\right] \\
&\quad + \mathbf{e}_0\mathbf{e}_3\mathbf{e}_1\left[-\left(\mathrm{div}\frac{\mathbf{E}_t}{c}\right)B_{sy} + \frac{1}{c^2}\frac{\partial E_{tx}}{\partial t}E_{sy} - \left(\mathrm{curl}\mathbf{B}_t\right)_x\frac{E_{sy}}{c}\right. \\
&\quad \left. - \frac{1}{c^2}\frac{\partial E_{tz}}{\partial t}E_{sx} + \left(\mathrm{curl}\mathbf{B}_t\right)_z\frac{E_{sx}}{c}\right] \\
&\quad + \mathbf{e}_0\mathbf{e}_2\mathbf{e}_3\left[-\left(\mathrm{div}\frac{\mathbf{E}_t}{c}\right)B_{sx} - \frac{1}{c^2}\frac{\partial E_{ty}}{\partial t}E_{sz} + \left(\mathrm{curl}\mathbf{B}_t\right)_y\frac{E_{sz}}{c}\right. \\
&\quad \left. + \frac{1}{c^2}\frac{\partial E_{tz}}{\partial t}E_{sy} - \left(\mathrm{curl}\mathbf{B}_t\right)_z\frac{E_{sy}}{c}\right] \\
&\quad + \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\left[\frac{1}{c^2}\frac{\partial E_{tx}}{\partial t}B_{sx} - \left(\mathrm{curl}\mathbf{B}_t\right)_x B_{sx} + \frac{1}{c^2}\frac{\partial E_{ty}}{\partial t}B_{sy}\right. \\
&\quad \left. - \left(\mathrm{curl}\mathbf{B}_t\right)_y B_{sy} + \frac{1}{c^2}\frac{\partial E_{tz}}{\partial t}B_{sz} - \left(\mathrm{curl}\mathbf{B}_t\right)_z B_{sz}\right]\Bigg\}\frac{\left(q_s q_t \overset{\text{biv}}{-} q_t q_s\right)}{2} \qquad (14.61)
\end{aligned}$$

which is Eq.(14.62) below. Then interchange s and t in (14.62) to obtain (14.63). Substitute $\mathbf{v}_s$ for $\mathbf{v}_t$ in (14.62) to obtain (14.64). Then substitute $\mathbf{F}_t$ for $\mathbf{F}_s$ in (14.62) to obtain (14.65).

$$\left(\mathbf{F}_s \overset{\text{triv}}{\wedge} \mathbf{V}_t\right)(q_s q_t - q_t q_s)/2$$
$$= \mathbf{e}_5 \left\{ \mathbf{e}_0 \left[-\frac{1}{c^2}\left(\frac{\partial \mathbf{E}_t}{\partial t}\right)\cdot \mathbf{B}_s + (\text{curl}\mathbf{B}_t)\cdot \mathbf{B}_s \right] + \left(\text{div}\frac{\mathbf{E}_t}{c}\right)\mathbf{B}_s + \frac{1}{c^3}\left(\frac{\partial \mathbf{E}_t}{\partial t}\right)\times \mathbf{E}_s - (\text{curl}\mathbf{B}_t)\times \mathbf{E}_s \right\}(q_s q_t - q_t q_s)/2 \quad (14.62)$$

$$(\mathbf{F}_t \wedge \mathbf{V}_s)(q_t q_s - q_s q_t)/2$$
$$= \mathbf{e}_5 \left\{ \mathbf{e}_0 \left[-\frac{1}{c^2}\left(\frac{\partial \mathbf{E}_s}{\partial t}\right)\cdot \mathbf{B}_t + (\text{curl}\mathbf{B}_s)\cdot \mathbf{B}_t \right] + \left(\text{div}\frac{\mathbf{E}_s}{c}\right)\mathbf{B}_t + \frac{1}{c^3}\left(\frac{\partial \mathbf{E}_s}{\partial t}\right)\times \mathbf{E}_t - (\text{curl}\mathbf{B}_s)\times \mathbf{E}_t \right\}(q_t q_s - q_s q_t)/2 \quad (14.63)$$

$$(\mathbf{F}_s \wedge \mathbf{V}_s)(0)$$
$$= \mathbf{e}_5 \left\{ \mathbf{e}_0 \left[-\frac{1}{c^2}\left(\frac{\partial \mathbf{E}_s}{\partial t}\right)\cdot \mathbf{B}_s + (\text{curl}\mathbf{B}_s)\cdot \mathbf{B}_s \right] + \left(\text{div}\frac{\mathbf{E}_s}{c}\right)\mathbf{B}_s + \frac{1}{c^3}\left(\frac{\partial \mathbf{E}_s}{\partial t}\right)\times \mathbf{E}_s - (\text{curl}\mathbf{B}_s)\times \mathbf{E}_s \right\}(q_s q_s - q_s q_s)/2 \quad (14.64)$$

$$(\mathbf{F}_t \wedge \mathbf{V}_t)(0)$$
$$= \mathbf{e}_5 \left\{ \mathbf{e}_0 \left[-\frac{1}{c^2}\left(\frac{\partial \mathbf{E}_t}{\partial t}\right)\cdot \mathbf{B}_t + (\text{curl}\mathbf{B}_t)\cdot \mathbf{B}_t \right] + \left(\text{div}\frac{\mathbf{E}_t}{c}\right)\mathbf{B}_t + \frac{1}{c^3}\left(\frac{\partial \mathbf{E}_t}{\partial t}\right)\times \mathbf{E}_t - (\text{curl}\mathbf{B}_t)\times \mathbf{E}_t \right\}(q_t q_t - q_t q_t)/2 \quad (14.65)$$

As we have seen, the term

$$\mathbf{F}_s \bullet \mathbf{V}_t \quad (14.66)$$

contributes to the indirect force and, therefore, describes a field (as summarized at the beginning of this section).

All of the terms in Eqs. (14.62, 14.63, 14.64, and 14.65) are zero. Therefore, the non-commutative terms, that is, those containing the factors $q_s q_t - q_t q_s$, etc., do not describe a field. Thus we conclude that the combination describing the field of a trivector:

$$\mathbf{F}_s \wedge \mathbf{V}_t \quad (14.67)$$

vanishes identically and therefore does not exist. Trivectors only interact via the direct force, that is, one-on-one. They cannot radiate energy. It could be called a dark force.

The square of Eq. (14.67) is:

$$(\mathbf{F}_s \wedge \mathbf{V}_t)(\mathbf{F}_s \wedge \mathbf{V}_t) \quad (14.68)$$

The preceding results show that a trivector, regardless of further details, is invisible. Its field is zero. To determine its complete structure, we need to examine its property as a direct force. We will show that the trivector force between masses is repulsive, and therefore, the result is negative gravity and can account for an accelerated expansion of galaxies and of the Universe as a whole. See Chapter 15 for a different treatment that shows the natural appearance of both trivector and vector forces.